MAT 162—Exam #1—2/20/15

Name: Solutions

Show all work using correct mathematical notation. Calculators are not allowed.

1. (15 points) Find the average value of the function $f(x) = \sqrt{x}$ on the interval $[1, 4]$.

$$f_{ave} = \frac{1}{4-1} \int_1^4 \sqrt{x}\, dx$$

$$= \frac{1}{3} \cdot \frac{2}{3} x^{3/2} \Big|_1^4$$

$$= \frac{2}{9} \left(4^{3/2} - 1 \right)$$

$$= \frac{14}{9}$$

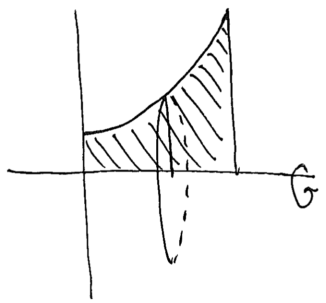
2. (10 points) Set up (**but do not evaluate**) a definite integral that gives the length of the curve $y = \sin(2x)$ from $x = 0$ to $x = \pi$.

$$f(x) = \sin(2x) \quad \Rightarrow \quad f'(x) = 2\cos(2x)$$

$$L = \int_0^{\pi} \sqrt{1 + (2\cos(2x))^2}\, dx$$

$$= \int_0^{\pi} \sqrt{1 + 4\cos^2(2x)}\, dx$$

3. (15 points) Find the volume of the solid obtained by revolving the region bounded by the curve $y = e^{3x}$ and the lines $x = 0$, $y = 0$, and $x = 1$ about the x-axis.



Disk radius $R(x) = e^{3x}$

Cross-sectional area $A(x) = \pi (e^{3x})^2 = \pi e^{6x}$

$$V = \int_0^1 \pi e^{6x}\, dx$$

$$= \frac{\pi}{6} e^{6x} \Big|_0^1$$

$$= \frac{\pi}{6} (e^6 - 1)$$

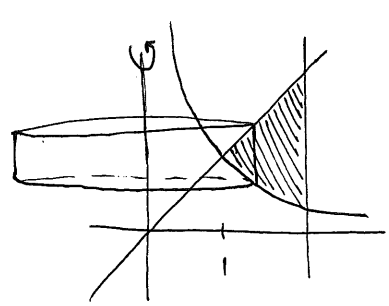
4. (15 points) Set up (**but do not evaluate**) definite integrals that give the volumes of the solids obtained by revolving the region bounded by the curves $y = 1/x$, $y = x$, and $x = 2$ about the given axes. In each case, show a representative disk, washer, or shell on the sketch provided.

(a) the y-axis

Shell radius : $r(x) = x$

Shell height : $h(x) = x - \frac{1}{x}$

$$V = \int_1^2 2\pi x \left(x - \frac{1}{x}\right) dx$$



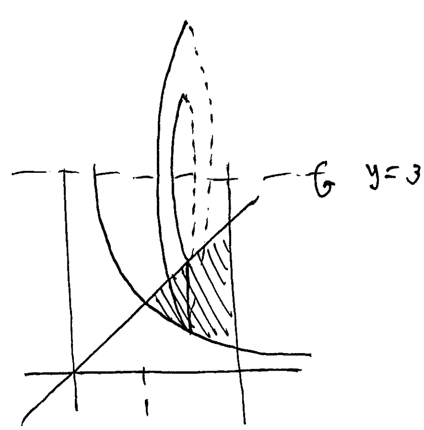
(b) the line $y = 3$

Washers :

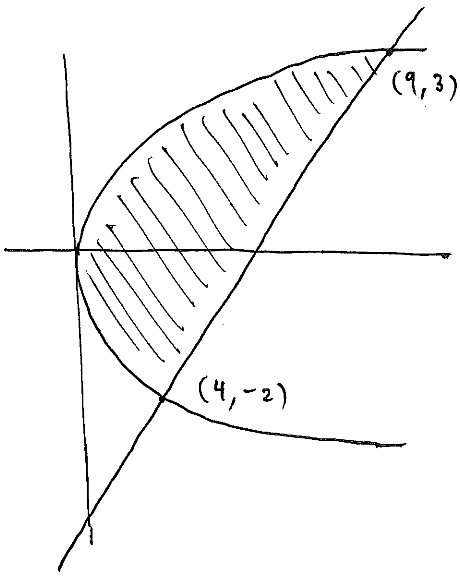
Outer radius : $R(x) = 3 - \frac{1}{x}$

Inner radius : $r(x) = 3 - x$

$$V = \int_1^2 \pi \left[\left(3 - \frac{1}{x}\right)^2 - (3 - x)^2 \right] dx$$



5. (15 points) Sketch the region bounded by the line $y = x - 6$ and the parabola $x = y^2$, and label the points of intersection. Then express the area of the region using one or more definite integrals. **Do not evaluate the integral(s).**



$$y^2 = y + 6 \iff y^2 - y - 6 = 0$$

$$\iff (y-3)(y+2) = 0$$

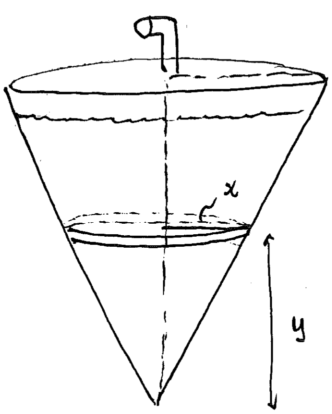
$$A = \int_{-2}^{3} (y + 6 - y^2)\, dy$$

OR

$$A = \int_0^4 2\sqrt{x}\, dx + \int_4^9 (\sqrt{x} - x + 6)\, dx$$

6. (15 points) A conical tank of radius 7 meters and height 10 meters is filled to a height of 9 meters with water, which weighs 9800 N/m^3. Water is to be pumped out through a spout that extends 2 meters above the tank's top.

(a) Find the weight of a slice of thickness Δy located at y meters from the bottom of the tank. Your answer should be expressed in terms of the variable y, as labeled in the diagram.



Similar Δ's : $\frac{x}{y} = \frac{7}{10} \Rightarrow x = \frac{7}{10} y$

Weight of slice $= 9800 \cdot \pi x^2 \, \Delta y$

$= 9800 \, \pi \left(\frac{7}{10} y\right)^2 \Delta y$

(b) Find the distance moved by the slice discussed in part (a) to reach the top of the spout.

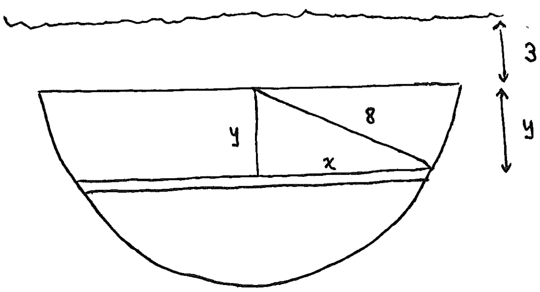
$12 - y$

(c) Set up (**but do not evaluate**) a definite integral that gives the total work required to empty the tank.

$$W = \int_0^9 9800\pi \left(\frac{7}{10} y\right)^2 (12 - y)\, dy$$

7. (15 points) A semi-circular plate of radius 8 feet is submerged in water, which weighs 62.4 lb/ft^3. The diameter of the plate lies 3 feet below the surface.

(a) Find the area of the strip of thickness Δy located at y feet below the top of the plate. Your answer should be expressed in terms of the variable y, as labeled in the diagram.



Pythagorean Theorem $\Rightarrow$

$$x^2 + y^2 = 64$$

$$\Rightarrow \quad x = \sqrt{64 - y^2}$$

$$\text{Area of strip} = 2x \; \Delta y$$

$$= 2\sqrt{64 - y^2} \; \Delta y$$

(b) Find the pressure along the strip discussed in part (a).

$$62.4 \; (y + 3)$$

(c) Set up (**but do not evaluate**) a definite integral that gives the hydrostatic force on the plate.

$$F = \int_0^8 62.4 \, (y+3) \cdot 2\sqrt{64 - y^2} \; dy$$