SOME MORE IDENTITIES OF KANADE–RUSSELL TYPE DERIVED USING ROSENGREN'S METHOD

JAMES MC LAUGHLIN

Abstract. In the present paper we consider some variations and generalizations of the multi-sum to single-sum transformation recently used by Rosengren in his proof of the Kanade–Russell identities.

These general transformations are then used to prove a number of identities equating multi-sums and infinite products or multi-sums and infinite product × a false theta series. Examples include the following:

$$\sum_{j,k,p,r=0}^{\infty} \frac{(-1)^{j+k} q^{(2j+k-p+r)^2/2+k(k+4)/2+3j-p/2+3r/2}(-q;q)_r}{(q^2;q^2)_j (q;q)_k (q;q)_p (q;q)_r} = 2\frac{(-q;q^2)_\infty (-q^2,-q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty}.$$

Let

$$Q(i,j,k,l,p) := \frac{1}{2}(i+6j+4k+2l-p)(i+6j+4k+2l-p-1) + 2k(k-1)+l(l-1)+3i+15j+14k+5l-2p.$$

Then

$$\sum_{i,j,k,l,p=0}^{\infty} \frac{(-1)^{l+k} q^{Q(i,j,k,l,p)}}{(q;q)_i (q^6;q^6)_j (q^4;q^4)_k (q^2;q^2)_l (q;q)_p} = \frac{2(-q;q)_\infty^2}{q\,(q^3;q^6)_\infty (q^4;q^4)_\infty}\left(1+\sum_{r=1}^{\infty}\left(q^{9r^2+6r}-q^{9r^2-6r}\right)\right).$$

$$\sum_{j,k,p=0}^{\infty} (-1)^k \frac{q^{(3j+2k-p)(3j+2k-p-1)/2+k(k-1)-p+6j+6k}}{(q^3;q^3)_j (q^2;q^2)_k (q;q)_p} = \frac{(-1;q)_\infty (q^{18};q^{18})_\infty}{(q^3;q^3)_\infty (q^9;q^{18})_\infty}.$$

1. Introduction

In the present paper we consider some variations and generalizations of the multi-sum to single-sum transformation used by Rosengren [22] in his proof of the Kanade–Russell identities.

Date: April 14, 2022.

2000 *Mathematics Subject Classification.* Primary: 33D15. Secondary:.

Key words and phrases. Basic Hypergeometric Series, q-series, partitions, q-products, multi-sum identities, Kanade–Russell identities, Rogers–Ramanujan identities, identities of Rogers–Ramanujan–Slater type, Capparelli identities .

These identities were conjectured by Kanade and Russell in [13], and were the result of a search conducted by them for Rogers–Ramanujan-type identities related to level 2 characters of $A_9^{(2)}$. They may be concisely stated using the notation of Rosengren [22]. Define

$$F(u,v,w) := \sum_{i,j,k=0}^{\infty} \frac{(-1)^k q^{3k(k-1)+(i+2j+3k)(i+2j+3k-1)} u^i v^j w^k}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}. \tag{1.1}$$

The identities conjectured by Kanade and Russell may be stated as follows:

$$F(q,1,q^3) = \frac{(q^3;q^{12})_\infty}{(q,q^2;q^4)_\infty}, \tag{1.2}$$

$$F(q^2,q^4,q^9) = \frac{(q^9;q^{12})_\infty}{(q^2,q^3;q^4)_\infty}, \tag{1.3}$$

$$F(q^4,q^6,q^{15}) = \frac{1}{(q^4,q^5,q^6,q^7,q^8;q^{12})_\infty}, \tag{1.4}$$

$$F(q,q^6,q^9) = \frac{1}{(q,q^4,q^6,q^8,q^{11};q^{12})_\infty}, \tag{1.5}$$

$$F(q^2,q^2,q^9) = \frac{(q^6;q^{12})_\infty}{(q^2,q^3,q^4;q^6)_\infty}, \tag{1.6}$$

$$F(q^3,q^5,q^{12}) = \frac{1}{(q^3;q^4)_\infty (q^4,q^5;q^{12})_\infty}, \tag{1.7}$$

$$F(q,q^3,q^6) = \frac{1}{(q;q^4)_\infty (q^4,q^{11};q^{12})_\infty}, \tag{1.8}$$

$$F(q,q,q^6) = \frac{1}{(q^3;q^4)_\infty (q,q^8;q^{12})_\infty}, \tag{1.9}$$

$$F(q^2,q^{-1},q^6) = \frac{1}{(q;q^4)_\infty (q^7,q^8;q^{12})_\infty}. \tag{1.10}$$

Kanade and Russell [13] also gave a combinatorial interpretation of each of these analytic identities in terms of integer partitions. Kurşungöz [16] gave analytic sum sides (generating functions) for six of the Kanade–Russell conjectures. Identities (1.2) - (1.6) were proved by Bringmann, Jennings–Shaffer and Mahlburg in [7], where they also proved two other conjectured identities of Kanade and Russell from [12].

A number of similar identities are stated elsewhere. Possibly two of the most widely known are the Capparelli identities, which were stated by Capparelli as conjectures in his thesis [8], not as analytic identities but as combinatorial identities involving certain types of restricted integer partitions. The first identity was first proved by Andrews [3], and Lie theoretic proofs of both conjectures were given by Tamba and Xie [28] and by Capparelli [9]. It is worth noting that the Capparelli identities arose from a study of level 3 standard modules for $A_2^{(2)}$, and many of the other identities mentioned here also arose through connections with affine Lie algebras.

Analytic versions of both Capparelli identities first appeared in the paper by Kanade and Russell [13], and were also given shortly afterwards by Kurşungöz [15], where they were derived independently by different methods, and with a slightly different sum side for the first identity. The versions stated by Kanade and Russell [13] were the following:

$$\sum_{i,j=0}^{\infty} \frac{q^{2i^2+6ij+6j^2}}{(q;q)_i\,(q^3;q^3)_j} = \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty},$$

$$\sum_{i,j,k,l=0}^{\infty} \frac{q^{\frac{3(i+j+k+2l)(i+j+k+2l-1)}{2}+3i+5j+\frac{3j(j-1)}{2}+k+\frac{3k(k-1)}{2}+3l}}{(q^3;q^3)_i\,(q^3;q^3)_j\,(q^3;q^3)_k\,(q^6;q^6)_l}$$
$$= \sum_{i,j=0}^{\infty} \frac{q^{2i^2+6ij+6j^2+i}}{(q;q)_i\,(q^3;q^3)_j} + \sum_{i,j=0}^{\infty} \frac{q^{2i^2+6ij+6j^2+4i+6j+1}}{(q;q)_i\,(q^3;q^3)_j}$$
$$= \left(-q,-q^3,-q^5,-q^6;q^6\right)_\infty .$$

More recently, Takigiku and Tsuchioka [26] give similar series - product identities for the principal characters of the level 5 and 7 standard modules of the affine Lie algebra $A_2^{(2)}$. An example of one of their identities from [26] is the following:

$$\sum_{i,j,k,\ell=0}^{\infty} (-1)^k \frac{q^{\binom{i}{2}+2\binom{j}{2}+2\binom{k}{2}+8\binom{\ell}{2}+ij+ik+2i\ell+4jk+4j\ell+4k\ell+i+3j+k+6\ell}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k(q^4;q^4)_l}$$
$$= \frac{1}{[q^2,q^3,q^4,q^5,q^6,q^7;q^{20}]_\infty}. \quad (1.11)$$

Here the notation (employed by the authors in [26])

$$[x;q]_\infty = (x,q/x;q)_\infty, \quad [a_1,\cdots,a_k;q]_\infty = [a_1;q]_\infty \cdots [a_k;q]_\infty$$

is used. In [27], Takigiku and Tsuchioka also prove three conjectures of Nandi (which arose from his study of the standard modules level 4 for the affine Lie algebra $A_2^{(2)}$ using a twisted vertex operator construction) from his thesis. Here is an example of one of those modulo 14 identities:

$$\sum_{i,k=0}^{\infty} (-1)^k \frac{q^{\binom{i}{2}+2\binom{k}{2}+2ik+i+k}}{(q;q)_i(q^2;q^2)_k} = \frac{1}{[q^2,q^3,q^4;q^{14}]_\infty}. \quad (1.12)$$

In [6], Berkovich and Uncu prove the following identity:

$$\sum_{m,n=0}^{\infty} \frac{q^{2m^2+6mn+6n^2-2m-3n}}{(q;q)_m(q^3;q^3)_n}$$
$$= (-q^2,-q^4;q^6)_\infty(-q^3;q^3)_\infty + (-q,-q^5;q^6)_\infty(-q^3;q^3)_\infty. \quad (1.13)$$

Note that the right side is the sum of the products in the analytic versions of the Capparelli identities [8].

Kanade and Russell have continued the study of the affine Lie algebra $A_2^{(2)}$ in their recent (2022) paper [14], where their results include producing Andrews–Gordon-type identities for standard modules levels 2 through 7. An example of one of their identities from that paper is the following (related to level 2):

$$\sum_{j_1\geq j_3\geq 0} \frac{(-1)^{j_1+j_3} q^{j_3^2-j_3+\binom{j_1}{2}+\binom{j_1-j_3}{2}}(-q;q)_{j_3}}{(-q;q)_{j_1}(q;q)_{j_1-j_3}(q;q)_{2j_3}} = \frac{(q^2,q^3,q^5;q^5)_\infty(q,q^9;q^{10})_\infty}{(q;q)_\infty}. \tag{1.14}$$

Since several authors refer to the series side of identities of this type as "Andrews–Gordon type series", we recall the theorem of Andrews [1], containing the referenced series, and which gave a generalization of the Rogers–Ramanujan identities.

Theorem 1.1 (Andrews, [1])**.** *Let $1 \leq i \leq k$ be integers; then*

$$\sum_{n_1,\cdots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \frac{(q^i,q^{2k+1-i},q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}. \tag{1.15}$$

where $N_j = n_j + \cdots + n_{k-1}$.

Note that the Rogers–Ramanujan identities follow from the cases $k = 2$ and $i = 1, 2$. One observable difference between the identities listed above and (1.15) is the presence of q-products with different moduli in the denominators of the series sides in the former. The series side in some of the series-product identities in Section 4 of the present paper are like the series in (1.15), in that the moduli in the q-products in the denominators *are* all the same (see Example 7).

In the present paper we prove a number of general transformations from a multi-sum to a single sum, such as

$$\begin{aligned}
\sum_{i,p,j_1,\dots j_v,k_1,\dots,k_s=0}^{\infty} & (-1)^{i+p+n_1j_1+\cdots+n_vj_v+(m_1+1)k_1+(m_s+1)k_s} \\
&\times \frac{q^{l(l-1)/2+\sum_{t=1}^s m_tk_t(k_t-1)/2}u^id^p\prod_{t=1}^s a_t^{k_t}\prod_{t=1}^v c_t^{j_t}}{(q;q)_t(q;q)_p\prod_{t=1}^v(q^{n_t};q^{n_t})_{j_t}\prod_{t=1}^s(q^{m_t};q^{m_t})_{k_t}} \\
&= \frac{(a_1d^{m_1};q^{m_1})_\infty\dots(a_sd^{m_s};q^{m_s})_\infty(qd,1/d;q)_\infty}{(c_1d^{n_1};q^{n_1})_\infty\dots(c_vd^{n_v};q^{n_v})_\infty(ud;q)_\infty} \\
&\qquad\times\sum_{k=0}^{\infty}\frac{(c_1d^{n_1};q^{n_1})_k\dots(c_vd^{n_v};q^{n_v})_k(ud;q)_k}{(a_1d^{m_1};q^{m_1})_k\dots(a_sd^{m_s};q^{m_s})_k(q;q)_k}\left(\frac{1}{d}\right)^k,
\end{aligned}$$

where

$$l = i - p + n_1j_1 + \cdots + n_vj_v + m_1k_1 + \cdots + m_sk_s$$

and the integer parameters $m_1, \dots, m_s, n_1, \dots, n_v$ satisfy

$$m_1 + \cdots + m_s = n_1 + \cdots + n_v.$$

These general transformations are then used to prove some multi-sum to infinite product (or in some cases, infinite product $\times$ false theta series) identities. Several of these identities have the flavour of the Kanade–Russell identities. Some examples are the following.

$$\sum_{j,k,p,r=0}^{\infty} \frac{(-1)^{j+k} q^{(2j+k-p+r)^2/2+k(k+4)/2+3j-p/2+3r/2}(-q;q)_r}{(q^2;q^2)_j (q;q)_k (q;q)_p (q;q)_r} = 2\frac{(-q;q^2)_\infty (-q^2,-q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty}.$$

Let

$$Q(i,j,k,l,p) := \frac{1}{2}(i+6j+4k+2l-p)(i+6j+4k+2l-p-1) + 2k(k-1)+l(l-1)+3i+15j+14k+5l-2p.$$

Then

$$\sum_{i,j,k,l,p=0}^{\infty} \frac{(-1)^{l+k} q^{Q(i,j,k,l,p)}}{(q;q)_i (q^6;q^6)_j (q^4;q^4)_k (q^2;q^2)_l (q;q)_p} = \frac{2(-q;q)_\infty^2}{q(q^3;q^6)_\infty (q^4;q^4)_\infty}\left(1+\sum_{r=1}^{\infty}\left(q^{9r^2+6r}-q^{9r^2-6r}\right)\right).$$

$$\sum_{j,k,p=0}^{\infty} (-1)^k \frac{q^{(3j+2k-p)(3j+2k-p-1)/2+k(k-1)-p+6j+6k}}{(q^3;q^3)_j (q^2;q^2)_k (q;q)_p} = \frac{(-1;q)_\infty (q^{18};q^{18})_\infty}{(q^3;q^3)_\infty (q^9;q^{18})_\infty}.$$

2. Some General Transformations

Before stating and proving some general transformations, we recall the statement of the q-binomial theorem and two special cases of it (see, for example, [4], equations (2.2.1), (2.2.5) and (2.2.6)).

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},\ |z|<1,\ |q|<1. \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q;q)_n} = (z;q)_\infty,\ |q|<1. \tag{2.3}$$

The following is a special case of a statement in [10, pages 125 - 127], which follows as a result of applying Cauchy's residue theorem.

Proposition 2.1 ([10], pages 125 - 127)**.** *Let* $|q|<1$ *and define*

$$P(z):=\frac{(a_1z,\dots,a_Az,qz,1/z;q)_\infty}{(c_1z,\dots,c_Az,uz,d/z;q)_\infty}. \tag{2.4}$$

Let the contour K *be a deformation of the positively oriented unit circle so that the poles of* $1/(c_1z,\dots,c_Az,uz;q)_\infty$ *lie outside* K *and the poles of* $1/(d/z;q)_\infty$ *lie inside* K.

Let

$$I:=\int_K P(z)\frac{dz}{2\pi iz}. \tag{2.5}$$

If $|1/d|<1$, *then*

$$I=\frac{(a_1d,\dots,a_Ad,qd,1/d;q)_\infty}{(q,c_1d,\dots,c_Ad,ud;q)_\infty}\sum_{k=0}^{\infty}\frac{(c_1d,\dots,c_Ad,ud;q)_k}{(a_1d,\dots,a_Ad,q;q)_k}\left(\frac{1}{d}\right)^k. \tag{2.6}$$

Sketch of proof (see [10], *pages 125 - 127 for details).* Consider the region whose outer boundary is the curve K and whose inner boundary is the circle C_N defined by the equation $|z|=\delta|q|^N$ with N sufficiently large so that the circle defined by this equation is entirely inside the curve K. The parameter δ is chosen so that $\delta|q|^N\neq d|q|^n$ for any integer n (so that the circle $|z|=\delta|q|^N$ does not pass through any of the poles of $1/(d/z;q)_\infty$, and this is also true if the integer N in the equation $|z|=\delta|q|^N$ is replaced with any integer $m>N$).

It can be shown (see [10, page 126] for details) that

$$\lim_{N\to\infty}\int_{C_N}P(z)\frac{dz}{2\pi iz}=0 \tag{2.7}$$

The residue at $z=dq^n$ of $P(z)/z$, after some simplification, is

$$\frac{(a_1d,\dots,a_Ad,qd,1/d;q)_\infty}{(q,c_1d,\dots,c_Ad,ud;q)_\infty}\frac{(c_1d,\dots,c_Ad,ud;q)_n}{(a_1d,\dots,a_Ad,dq;q)_n}\frac{(dq;q)_n}{(q;q)_n}\left(\frac{1}{d}\right)^n.$$

Then (2.6) follows from applying the Cauchy residue theorem in conjunction with (2.7). □

Next, continuing as Rosengren did, we can use the Cauchy integral formula to treat $(q;q)_\infty I$ as being equal to the constant term in the Laurent series expansion about 0 of $(q;q)_\infty P(z)$, and thus get the result in the next theorem.

Theorem 2.1. *Let* $\max\{|c_1d|,|c_2d|,\dots,|c_Ad|,|ud|,|1/d|\}<1$ *and let* I *be as at* (2.5). *Then*

$$(q;q)_\infty I=\sum_{i,p,j_1,\dots j_A,k_1,\dots,k_A=0}^{\infty}(-1)^{i+p+j_1+\cdots+j_A}\frac{q^{l(l-1)/2+\sum_{t=1}^{A}k_t(k_t-1)/2}u^id^p\prod_{t=1}^{A}a_t^{k_t}c_t^{j_t}}{(q;q)_i(q;q)_p\prod_{t=1}^{A}(q;q)_{j_t}(q;q)_{k_t}}, \tag{2.8}$$

where

$$l = i - p + j_1 + \cdots + j_A + k_1 + \cdots + k_A.$$

Proof. By (2.4), (2.2), (2.3) and (3.2)

$$(q;q)_\infty P(z) = \sum_{k_1=0}^{\infty} \frac{(-a_1 z)^{k_1} q^{k_1(k_1-1)/2}}{(q;q)_{k_1}} \cdots \sum_{k_A=0}^{\infty} \frac{(-a_A z)^{k_A} q^{k_A(k_A-1)/2}}{(q;q)_{k_A}}$$

$$\times \sum_{l=-\infty}^{\infty} q^{l(l-1)/2}(-1/z)^l \sum_{j_1=0}^{\infty} \frac{(c_1 z)^{j_1}}{(q;q)_{j_1}} \cdots \sum_{j_A=0}^{\infty} \frac{(c_A z)^{j_A}}{(q;q)_{j_A}} \sum_{i=0}^{\infty} \frac{(uz)^i}{(q;q)_i} \sum_{p=0}^{\infty} \frac{(d/z)^p}{(q;q)_p}.$$

The exponent of z in this multi-sum is

$$k_1 + \cdots + k_A + j_1 + \cdots + j_A + i - p - l,$$

and this exponent is zero if l has the value stated in the theorem, and the result follows. □

Upon equating the expressions for I in the two theorems, we get a multi-sum to single sum identity.

Theorem 2.2. *Let* $\max\{|c_1 d|, |c_2 d|, \ldots, |c_A d|, |ud|, |1/d|\} < 1$. *Then*

$$\sum_{i,p,j_1,\ldots j_A,k_1,\ldots,k_A=0}^{\infty} (-1)^{i+p+j_1+\cdots+j_A} \frac{q^{l(l-1)/2+\sum_{t=1}^{A} k_t(k_t-1)/2} u^i d^p \prod_{t=1}^{A} a_t^{k_t} c_t^{j_t}}{(q;q)_i (q;q)_p \prod_{t=1}^{A} (q;q)_{j_t} (q;q)_{k_t}}$$

$$= \frac{(a_1 d, \ldots, a_A d, qd, 1/d; q)_\infty}{(c_1 d, \ldots, c_A d, ud; q)_\infty} \sum_{k=0}^{\infty} \frac{(c_1 d, \ldots, c_A d, ud; q)_k}{(a_1 d, \ldots, a_A d, q; q)_k} \left(\frac{1}{d}\right)^k, \quad (2.9)$$

where

$$l = i - p + j_1 + \cdots + j_A + k_1 + \cdots + k_A.$$

Remark: If the value of some a_f is the same as the value of one of the c_g or of u, then the terms involving this common value cancel on the right side of (2.9) cancel, so this right side becomes independent of this common value. Thus the left side is also independent of this common value, indeed the coefficient of any non-zero power of it being identically zero. To see this, supposes $v := a_f = c_g$ (or u). Make a change of summation variables $(k_f, j_g) \to (k_f, r)$, where $r = k_f + j_g$ or $j_g = r - k_f$. The coefficient of v^r is a nested sum over the remaining variables, but it can be seen that the sum over k_f is

$$\sum_{k_f=0}^{r} \frac{(-1)^{k_f} q^{k_f(k_f-1)/2}}{(q;q)_{k_f} (q;q)_{r-k_f}}$$

$$= \frac{1}{(q;q)_r} \sum_{k_f=0}^{r} (-1)^{k_f} q^{k_f(k_f-1)/2} \begin{bmatrix} r \\ k_f \end{bmatrix} = \begin{cases} 1, & \text{if } r = 0 \\ 0, & \text{otherwise,} \end{cases} \quad (2.10)$$

by the $z = 1$ case of (3.1). Thus the only value of $r = k_f + j_g$ that contributes a non-zero term to the multi-sum is $r = 0$. The upshot of this is that if some

a_f is set equal to some c_g or u in (2.9), then these terms and the sums over the corresponding summation variables k_f and j_g may be eliminated from the multi-sum. A similar conclusion could have been reached by just letting $v \to 0$, which would not have interfered with the requirements of the contour integral.

By similar reasoning to that used in the proof of the previous theorem, it is easy to give a generalization of the result in that theorem.

Theorem 2.3. *Let* $\max\{|c_1 d^{n_1}|, |c_2 d^{n_2}|, \dots, |c_v d^{n_v}|, |ud|, |1/d|\} < 1$. *Let* $m_1, \dots, m_s, n_1, \dots, n_v$ *be integers such that*

$$m_1 + \cdots + m_s = n_1 + \cdots + n_v.$$

Then

$$\sum_{i,p,j_1,\dots j_v,k_1,\dots,k_s=0}^{\infty} (-1)^{i+p+n_1 j_1+\cdots+n_v j_v+(m_1+1)k_1+(m_s+1)k_s} \\ \times \frac{q^{l(l-1)/2+\sum_{t=1}^{s} m_t k_t(k_t-1)/2} u^i d^p \prod_{t=1}^{s} a_v^{k_t} \prod_{t=1}^{v} c_t^{j_t}}{(q;q)_t (q;q)_p \prod_{t=1}^{v}(q^{n_t};q^{n_t})_{j_t} \prod_{t=1}^{s}(q^{m_t};q^{m_t})_{k_t}} \\ = \frac{(a_1 d^{m_1};q^{m_1})_\infty \dots (a_s d^{m_s};q^{m_s})_\infty (qd, 1/d;q)_\infty}{(c_1 d^{n_1};q^{n_1})_\infty \dots (c_v d^{n_v};q^{n_v})_\infty (ud;q)_\infty} \\ \sum_{k=0}^{\infty} \frac{(c_1 d^{n_1};q^{n_1})_k \dots (c_v d^{n_v};q^{n_v})_k (ud;q)_k}{(a_1 d^{m_1};q^{m_1})_k \dots (a_s d^{m_s};q^{m_s})_k (q;q)_k} \left(\frac{1}{d}\right)^k, \tag{2.11}$$

where

$$l = i - p + n_1 j_1 + \cdots + n_v j_v + m_1 k_1 + \cdots + m_s k_s.$$

Proof. Both sides of (2.11) come from consideration of the product

$$P_2(z) := \frac{(a_1 z^{m_1};q^{m_1})_\infty \dots (a_s z^{m_s};q^{m_s})_\infty (qz, 1/z;q)_\infty}{(c_1 z^{n_1};q^{n_1})_\infty \dots (c_v z^{n_v};q^{n_v})_\infty (uz, d/z;q)_\infty}. \tag{2.12}$$

The left side is the constant term in the series expansion of $(q;q)_\infty P_2(z)$. The right side comes from applying Theorem 2.1 to $P_2(z)$, employing the elementary identities (where $\zeta = \exp(2\pi i/r)$ is a primitive r-th root of unity)

$$(ez^r;q^r)_\infty = \prod_{i=0}^{r-1} (\sqrt[r]{e}\zeta^i z;q)_\infty,$$

$$(ez^r;q^r)_k = \prod_{i=0}^{r-1} (\sqrt[r]{e}\zeta^i z;q)_k$$

first to $P_2(z)$ to write the all infinite products in the form $(x;q)_\infty$ so that Theorem 2.1 can be applied, and then to recombine finite- and infinite products on the right side of (2.6). □

Another variation is necessary in order to handle identities like the Kanade –Russell identities.

Theorem 2.4. *Let* $\max\{|c_1d^{n_1}|, |c_2d^{n_2}|, \ldots, |c_vd^{n_v}|, |ud|, |1/d|\} < 1$. *Let* $m_1, \ldots, m_s$, $n_1, \ldots, n_v$ *and* $r > 1$ *be integers such that*

$$m_1 + \cdots + m_s = n_1 + \cdots + n_v + r - 1.$$

Then

$$\sum_{i,p,j_1,\ldots j_v,k_1,\ldots,k_s=0}^{\infty} (-1)^{i+p+n_1j_1+\cdots+n_vj_v+(m_1+1)k_1+(m_s+1)k_s} \times \frac{q^{rl(l-1)/2+\sum_{t=1}^{s} rm_tk_t(k_t-1)/2} u^i d^p \prod_{t=1}^{s} a_t^{k_t} \prod_{t=1}^{v} c_t^{j_t}}{(q;q)_i (q^r;q^r)_p \prod_{t=1}^{v}(q^{rn_t};q^{rn_t})_{j_t} \prod_{t=1}^{s}(q^{rm_t};q^{rm_t})_{k_t}}$$
$$= \frac{(a_1d^{m_1};q^{rm_1})_\infty \ldots (a_sd^{m_s};q^{rm_s})_\infty (q^rd, 1/d;q^r)_\infty}{(c_1d^{n_1};q^{rn_1})_\infty \ldots (c_vd^{n_v};q^{rn_v})_\infty (ud;q)_\infty}$$
$$\sum_{k=0}^{\infty} \frac{(c_1d^{n_1};q^{rn_1})_k \ldots (c_vd^{n_v};q^{rn_v})_k (ud;q)_{kr}}{(a_1d^{m_1};q^{rm_1})_k \ldots (a_sd^{m_s};q^{rm_s})_k (q^r;q^r)_k} \left(\frac{1}{d}\right)^k, \quad (2.13)$$

where

$$l = i - p + n_1j_1 + \cdots + n_vj_v + m_1k_1 + \cdots + m_sk_s.$$

Proof. As in the previous theorem, each side of (2.13) comes from consideration of an infinite product, this time

$$P_3(z) := \frac{(a_1z^{m_1};q^{rm_1})_\infty \ldots (a_sz^{m_s};q^{rm_s})_\infty (q^rz, 1/z;q^r)_\infty}{(c_1z^{n_1};q^{rn_1})_\infty \ldots (c_vz^{n_v};q^{rn_v})_\infty (uz;q)_\infty (d/z;q^r)_\infty}.$$

The left side is the constant term in the series expansion of $(q;q)_\infty P_3(z)$. The proof of the right side is similar to the proof of the right side in Theorem 2.3, except with q^r instead of q. Also, in $P_3(z)$, $(uz;q)_\infty$ is written as $(uz, uzq, \ldots, uzq^{r-1};q^r)_\infty$, in the infinite product in front of the series on the right side, $(ud, udq, \ldots, udq^{r-1};q^r)_\infty$ is written as $(ud;q)_\infty$, and in the series on the right, $(ud, udq, \ldots, udq^{r-1};q^r)_k$ is written as $(ud;q)_{kr}$. □

Remark: Even though Theorem 2.4 is a generalization/extension of the previous two theorems, we find it simpler to build up to Theorem 2.4 through proving, in turn, Theorems 2.2 and 2.3, rather than trying to prove Theorem 2.4 from scratch.

We consider one further variation, where $(ez;q)_\infty/(fz;q)_\infty$ is expanded using the standard q-binomial theorem (2.1),

$$\frac{(ez;q)_\infty}{(fz;q)_\infty} = \sum_{n=0}^{\infty} \frac{(e/f;q)_n}{(q;q)_n}(fz)^n, \quad (2.14)$$

rather than using the special cases (2.2) and (2.3) to expand the numerator and denominator separately.

Let

$$P_4(z) := \frac{(a_1z, \ldots, a_Az, e_1z, \ldots, e_Bz, qz, 1/z;q)_\infty}{(c_1z, \ldots, c_Az, f_1z, \ldots, f_Bz, uz, d/z;q)_\infty}. \quad (2.15)$$

Let the contour K be a deformation of the positively oriented unit circle so that the poles of $1/(c_1 z, \dots, c_A z, f_1 z, \dots, f_B z, uz; q)_\infty$ lie outside K and the poles of $1/(d/z; q)_\infty$ lie inside K.

Let

$$I_4 := \int_K P_4(z) \frac{dz}{2\pi i z}. \tag{2.16}$$

Let

Theorem 2.5. *Let* $\max\{|c_1 d|, \dots, |c_A d|, |f_1 d|, \dots, |f_B d|, |ud|, |1/d|\} < 1$ *and let* I_4 *be as at* (2.16). *Then*

$$(q;q)_\infty I_4 = \sum_{i,p,j_1,\dots j_A,k_1,\dots,k_A,r_1,\dots r_B=0}^{\infty} (-1)^{i+p+j_1+\dots+j_A+r_1+\dots+r_B} \times \frac{q^{l(l-1)/2+\sum_{t=1}^{A} k_t(k_t-1)/2} u^i d^p \prod_{t=1}^{A} a_t^{k_t} c_t^{j_t} \prod_{t=1}^{B} (e_t/f_t; q)_{r_t} f_t^{r_t}}{(q;q)_i (q;q)_p \prod_{t=1}^{A} (q;q)_{j_t} (q;q)_{k_t} \prod_{t=1}^{B} (q;q)_{r_t}}, \tag{2.17}$$

where

$$l = i - p + j_1 + \dots + j_A + k_1 + \dots + k_A + r_1 + \dots + r_B.$$

Proof. The proof is similar to the proof of Theorem 2.1, except that there is an extra set of series expansions,

$$\sum_{r_1=0}^{\infty} \frac{(e_1/f_1; q)_{r_1} (f_1 z)^{r_1}}{(q;q)_{r_1}} \cdots \sum_{r_B=0}^{\infty} \frac{(e_B/f_B; q)_{r_B} (f_B z)^{r_B}}{(q;q)_{r_B}}.$$

The details are omitted. □

The equivalent of Theorem 2.2 follows easily, by combining the result in the previous theorem with a slightly amended version of Theorem 2.1.

Theorem 2.6. *Let* $\max\{|c_1 d|, \dots, |c_A d|, |f_1 d|, \dots, |f_B d|, |ud|, |1/d|\} < 1$. *Then*

$$\begin{aligned} &\sum_{i,p,j_1,\dots j_A,k_1,\dots,k_A,r_1,\dots r_B=0}^{\infty} (-1)^{i+p+j_1+\dots+j_A+r_1+\dots+r_B} \\ &\times \frac{q^{l(l-1)/2+\sum_{t=1}^{A} k_t(k_t-1)/2} u^i d^p \prod_{t=1}^{A} a_t^{k_t} c_t^{j_t} \prod_{t=1}^{B} (e_t/f_t; q)_{r_t} f_t^{r_t}}{(q;q)_i (q;q)_p \prod_{t=1}^{A} (q;q)_{j_t} (q;q)_{k_t} \prod_{t=1}^{B} (q;q)_{r_t}} \\ &= \frac{(a_1 d, \dots, a_A d, e_1 d, \dots, e_B d, qd, 1/d; q)_\infty}{(c_1 d, \dots, c_A d, f_1 d, \dots, f_B d, ud; q)_\infty} \\ &\times \sum_{k=0}^{\infty} \frac{(c_1 d, \dots, c_A d, f_1 d, \dots, f_B d, ud; q)_k}{(a_1 d, \dots, a_A d, e_1 d, \dots, e_B d, q; q)_k} \left(\frac{1}{d}\right)^k, \end{aligned} \tag{2.18}$$

where

$$l = i - p + j_1 + \dots + j_A + k_1 + \dots + k_A + r_1 + \dots + r_B.$$

Proof. The proof is essentially the same as that of Theorem 2.2, so the details are omitted. □

It is also possible to give slight generalizations of Theorem 2.6, as was done with Theorem 2.2 in Theorems 2.3 and 2.4. We state one such generalization and give an application.

Theorem 2.7. *Let $m_1,\dots, m_s$, $n_1, \dots ,n_v$ be positive integers such that*

$$m_1 + \cdots + m_s = n_1 + \cdots + n_v.$$

Let

$$\max\{|c_1 d^{n_1}|, |c_2 d^{n_2}|, \dots, |c_v d^{n_v}|, |f_1 d|, \dots, |f_B d|, |ud|, |1/d|\} < 1.$$

Then

$$\begin{aligned}
&\sum_{i,p,j_1,\dots j_v,k_1,\dots,k_s=0}^{\infty} (-1)^{i+p+n_1j_1+\cdots+n_vj_v+(m_1+1)k_1+(m_s+1)k_s+r_1+\cdots+r_B}\\
&\times \frac{q^{l(l-1)/2+\sum_{t=1}^{s} m_t k_t(k_t-1)/2} u^i d^p \prod_{t=1}^{s} a_t^{k_t} \prod_{t=1}^{v} c_t^{j_t} \prod_{t=1}^{B} (e_t/f_t;q)_{r_t} f_t^{r_t}}{(q;q)_i (q;q)_p \prod_{t=1}^{v} (q^{n_t};q^{n_t})_{j_t} \prod_{t=1}^{s} (q^{m_t};q^{m_t})_{k_t} \prod_{t=1}^{B} (q;q)_{r_t}}\\
&= \frac{(a_1 d^{m_1};q^{m_1})_\infty \dots (a_s d^{m_s};q^{m_s})_\infty (qd,1/d;q)_\infty (e_1 d,\dots,e_B d;q)_\infty}{(c_1 d^{n_1};q^{n_1})_\infty \dots (c_v d^{n_v};q^{n_v})_\infty (ud;q)_\infty (f_1 d,\dots,f_B d;q)_\infty}\\
&\sum_{k=0}^{\infty} \frac{(c_1 d^{n_1};q^{n_1})_k \dots (c_v d^{n_v};q^{n_v})_k (ud;q)_k (f_1 d,\dots,f_B d;q)_k}{(a_1 d^{m_1};q^{m_1})_k \dots (a_s d^{m_s};q^{m_s})_k (q;q)_k (e_1 d,\dots,e_B d;q)_k} \left(\frac{1}{d}\right)^k, \qquad (2.19)
\end{aligned}$$

where

$$l = i - p + n_1 j_1 + \cdots + n_v j_v + m_1 k_1 + \cdots + m_s k_s + r_1 + \cdots + r_B.$$

Proof. The proof is essentially the same as the proof of Theorem 2.3, except that instead of considering the product $P_2(z)$ at (2.12), each side of (2.19) arises from consideration of the product

$$P_5(z) := P_2(z) \frac{(e_1 z,\dots,e_B z;q)_\infty}{(f_1 z,\dots,f_B z;q)_\infty}. \qquad (2.20)$$

The details are omitted. □

3. Some Examples That Follow from the general results

Before deriving some consequences of the transformations in the previous sections, we recall some other well-known series product identities.

$$\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} (-z)^n q^{n(n-1)/2} = (z;q)_N, \qquad (3.1)$$

$$\sum_{n=-\infty}^{\infty} (-1/z)^n q^{n(n-1)/2} = (1/z, zq, q; q)_\infty. \qquad (3.2)$$

$$\sum_{k=0}^{\infty}\frac{(A,B,C;q)_k}{(D,E,q;q)_k}\left(\frac{DE}{ABC}\right)^k$$
$$=\frac{(E/A,DE/BC;q)_\infty}{(E,DE/ABC;q)_\infty}\sum_{k=0}^{\infty}\frac{(A,D/B,D/C;q)_k}{(D,DE/BC,q;q)_k}\left(\frac{E}{A}\right)^k. \quad (3.3)$$

$$\sum_{k=0}^{\infty}\frac{(A,B;q)_k}{(Aq/B,q;q)_k}\left(\frac{-q}{B}\right)^k=\frac{(-q;q)_\infty(Aq,Aq^2/B^2;q^2)_\infty}{(Aq/B,-q/B;q)_\infty}. \quad (3.4)$$

Identities (3.1) (a special case of the q-binomial theorem) and (3.2) (the Jacobi triple product identity) may be found in the book of Andrews [4], being respectively, equations (3.3.6) and (2.2.10). Identities (3.3) - (3.4) above may be found in the book by Gasper and Rahman [10], being respectively, equations (III.9) in Appendix III and (II.9) in Appendix II.

It can be seen from the general transformations in the previous section that multi-sum- to single-sum identities are very plentiful, so instead we would like to give some multi-sum to infinite product/infinite product $\times$ false theta series identities. We give a number of applications of Theorems 2.3 and (2.4), some of which have something of the flavor of some of the Kanade–Russell identities. We need the following identities [18, Eqs. (3.11) - (3.13)]:

$$\sum_{n=1}^{\infty}\frac{(q;q^2)_n q^n}{(-q;q^2)_{n+1}}=1+\sum_{r=1}^{\infty}q^{8r^2}(q^{4r}-q^{-4r}), \quad (3.5)$$

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^n}{(q^2;q^2)_{n+1}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(-q^2,-q^{14},q^{16};q^{16})_\infty, \quad (3.6)$$

$$\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n(-q)^n}{(q^2;q^2)_{n+1}(q;q)_n}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\frac{(q^{18};q^{18})_\infty}{(q^9;q^{18})_\infty}. \quad (3.7)$$

To avoid repetition, we state here that the summation variables j_1 and k_1 arising from the various multi-sum to single-sum transformations stated above are replaced with j and k in the following examples (for the purpose of making the identities easier to read).

Example 1. *If* $|q|<1$*, then*

$$\sum_{i,j,k,p=0}^{\infty}(-1)^{j+k}\frac{q^{(i+2j+2k-p)(1+2j+2k-p-1)/2+k(k-1)+2i-p+3j+5k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k(q;q)_p}$$
$$=\frac{(-1;q^2)_\infty}{(q;q)_\infty}\left(1+\sum_{r=1}^{\infty}q^{8r^2}(q^{4r}-q^{-4r})\right). \quad (3.8)$$

Proof. In (2.11), set $s = t = 1$, $m_1 = n_1 = 2$, $a_1 = q^5$, $c_1 = -q^3$, $d = -1/q$ and $u = -q^2$, so that the left side of (2.11) becomes the left side of (3.8). That the right side of (2.11) becomes the right side of (3.8) follows upon making the indicated substitutions, effecting some elementary q-product manipulations, and noting that the right side of (3.5) is invariant under the substitution $q \to -q$ (and hence so also is the left side). □

The next example illustrates how one single-series-to-product identity can give rise to more than one multi-sum identity.

Example 2. *If* $|q| < 1$, *then*

$$\sum_{j,k,p=0}^{\infty} (-1)^k \frac{q^{(2j+k-p)(2j+k-p-1)+k(k-1)-p+6j+5k}}{(q^4;q^4)_j(q^2;q^2)_k(q^2;q^2)_p} = \frac{(-q;q^2)_\infty(-q^2,-q^{14},q^{16};q^{16})_\infty}{(-q^2;q^2)_\infty(q^4;q^4)_\infty}; \quad (3.9)$$

$$\sum_{i,j,k,p=0}^{\infty} (-1)^{j+k} \frac{q^{(i+2j+2k-p)(1+2j+2k-p-1)/2+k(k-1)+2i-p+4j+6k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k(q;q)_p} = 2\frac{(-q;q^2)_\infty(-q^2,-q^{14},q^{16};q^{16})_\infty}{(q;q)_\infty}. \quad (3.10)$$

Proof. For (3.9), in (2.13) set $r = 2$, $s = t = 1$, $m_1 = 1$, $n_1 = 2$, $a_1 = -q^5$, $c_1 = q^6$, $d = -1/q$ and $u = 0$ (so that the sum over i vanishes). Then (3.9) follows almost directly, with all that is necessary being to simplify the resulting series on the right side with some elementary q-product manipulations, and to then employ (3.6) (with q replaced with $-q$).

For (3.10), in (2.11) set $s = t = 1$, $m_1 = n_1 = 2$, $a_1 = q^6$, $c_1 = -q^4$, $d = -1/q$ and $u = -q^2$. The result follows once again after employing (3.6) (again with q replaced with $-q$). □

Example 3. *If* $|q| < 1$, *then*

$$\sum_{j,k,p=0}^{\infty} (-1)^k \frac{q^{(3j+2k-p)(3j+2k-p-1)/2+k(k-1)-p+6j+6k}}{(q^3;q^3)_j(q^2;q^2)_k(q;q)_p} = \frac{(-1;q)_\infty(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty(q^9;q^{18})_\infty}. \quad (3.11)$$

Proof. In Theorem 2.3, let $s = 2$ with $m_1 = 2$ and $m_1 = 1$, let $v = 1$ with $n_1 = 3$, and set $a_2 = u$, so that by the remarks preceding Theorem 2.3, a_2, u and the sums over i and k_2 vanish from the multi-sum. Next, set $a_1 = q^6$, $c_1 = -q^6$ and $d = -1/q$, replace j_1 with j and k_1 with k, so that $l = 3j + 2k - p$ and the left side of (2.11) becomes the left side of (3.11).

The right side of (2.11) becomes

$$\frac{(q^4;q^2)_\infty(-1,-q;q)_\infty}{(q^3;q^3)_\infty}\sum_{k=0}^{\infty}\frac{(q^3;q^3)_k(-q)^k}{(q^4;q^2)_k(q;q)_k},$$

and the result follows from (3.7), after some simplification □

Before proving the next multi-sum identity, it is necessary to prove the false theta identity in the following lemma.

Lemma 3.1. *If* $|q|<1$*, then*

$$\sum_{r=0}^{\infty}\frac{(-1)^r\left(q^3;q^6\right)_r q^{2r}}{\left(q^2;q^4\right)_{r+1}\left(q;q^2\right)_r}=1+\sum_{r=1}^{\infty}q^{9r^2}\left(q^{6r}-q^{-6r}\right). \tag{3.12}$$

Proof. This follows from inserting Slater's Bailey pair ([24, page 467]) with respect to $a=q$

$$\begin{aligned}\beta_n(q)&=\frac{(q^{3/2};q^3)_n q^n}{(q^2;q)_{2n}(q^{1/2};q)_n},\\ \alpha_{3r-1}(q)&=(-1)^r q^{(9r^2-6r)/2},\\ \alpha_{3r}(q)&=(-1)^r q^{(9r^2+6r)/2},\\ \alpha_{3r+1}(q)&=(-1)^{r+1}(q^{(9r^2+6r)/2}-q^{(9r^2+12r+3)/2})\end{aligned}$$

into the particular case of the Bailey transform (with respect to $a=q$)

$$\sum_{n=0}^{\infty}(aq;q^2)_n\,(-1)^n\,\beta_n=\frac{1}{(aq^2;q^2)_\infty(-1;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\,\alpha_n, \tag{3.13}$$

then making the replacement $q\to q^2$, and finally multiplying both sides by $1/(1-q^2)$. Note that the sum over the $(-1)^n\,\alpha_n$ on the right side of (3.13) (with the replacement $q\to q^2$) leads to twice the sum on the right side of (3.12). Then with all the indicated changes, the factor multiplying the series on the right side of (3.12) becomes $2/(q^2;q^4)_\infty(-1;q^2)_\infty=1$, giving the result. □

Example 4. *Let*

$$\begin{aligned}Q(i,j,k,l,p):=\frac{1}{2}&(i+6j+4k+2l-p)(i+6j+4k+2l-p-1)\\ &+2k(k-1)+l(l-1)+3i+15j+14k+5l-2p.\end{aligned}$$

If $|q|<1$*, then*

$$\begin{aligned}\sum_{i,j,k,l,p=0}^{\infty}&\frac{(-1)^{l+k}q^{Q(i,j,k,l,p)}}{(q;q)_i\,(q^6;q^6)_j\,(q^4;q^4)_k\,(q^2;q^2)_l(q;q)_p}\\ &=\frac{2(-q;q)_\infty^2}{q\,(q^3;q^6)_\infty\,(q^4;q^4)_\infty}\left(1+\sum_{r=1}^{\infty}\left(q^{9r^2+6r}-q^{9r^2-6r}\right)\right).\end{aligned} \tag{3.14}$$

Proof. In (2.11), let $s = 2$ with $m_1 = 4$ and $m_2 = 2$, $t = 1$ with $n_1 = 6$. Set $a_1 = q^{14}$, $a_2 = q^5$, $c_1 = q^{15}$, $d = -1/q^2$ and $u = -q^3$. After making the replacements $k_1 \to k$, $k_2 \to l$ and $j_1 \to j$, the left side of (2.11) becomes the left side of (3.14).

The right side of (2.11) transforms (without simplification) to

$$\frac{(q^6;q^4)_\infty(q;q^2)_\infty(-1/q,-q^2;q)_\infty}{(q^3;q^6)_\infty(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^3;q^6)_k(q;q)_k(-q^2)^k}{(q^6;q^4)_k(q;q^2)_k(q;q)_k},$$

which simplifies to the right side of (3.14) upon employing (3.12). □

Many of the details in the following example, a proof of one of the Kanade–Russell identities, parallel details in the proof of Rosengren [22], as might be expected, but we include this example to give an application of Theorem 2.4.

Example 5. *If* $|q| < 1$, *then*

$$\sum_{i,j,k=0}^{\infty}\frac{(-1)^k q^{3(k-1)k+(i+2j+3k)(i+2j+3k-1)+i+3k}}{(q;q)_i\,(q^4;q^4)_j\,(q^6;q^6)_k}=\frac{(q^3;q^{12})_\infty}{(q,q^2;q^4)_\infty}. \tag{3.15}$$

Proof. In (2.13), set $r = 2$, $s = t = 1$, $m_1 = 3$ and $n_1 = 2$ so that $l = i - p + 2j_1 + 3k_1$ and

$$\sum_{i,p,j_1,k_1=0}^{\infty}(-1)^{i+p}\frac{q^{(i-p+2j_1+3k_1)(i-p+2j_1+3k_1-1)+3k_1(k_1-1)}u^i d^p a_1^{k_1}c_1^{j_1}}{(q;q)_i(q^2;q^2)_p(q^4;q^4)_{j_1}(q^6;q^6)_{k_1}}$$
$$=\frac{(a_1d^3;q^6)_\infty(q^2d,1/d;q^2)_\infty}{(c_1d^2;q^4)_\infty(ud;q)_\infty}\sum_{k=0}^{\infty}\frac{(c_1d^2;q^4)_k(ud,udq;q^2)_k}{(a_1d^3;q^6)_k(q^2;q^2)_k}\left(\frac{1}{d}\right)^k, \tag{3.16}$$

after writing $(ud;q)_{2k} = (ud,udq;q^2)_k$.

Next, write

$$(c_1d^2;q^4)_k = (\sqrt{c_1}d,-\sqrt{c_1}d;q^2)_k,$$
$$(a_1d^3;q^6)_k = (\sqrt[3]{a_1}d,\,\omega\sqrt[3]{a_1}d,\,\omega^2\sqrt[3]{a_1}d;q^2)_k$$

where $\omega = \exp(2\pi i/3)$. Next, set $c_1 = 1$, $a_1 = -q^3$ and $u = -q$, so that after cancelling a factor of $(-qd;q^2)_k$, the series on the right side of (3.16) becomes

$$\sum_{k=0}^{\infty}\frac{(d,-d,-dq^2;q^2)_k}{(-\omega qd,-\omega^2qd,q^2;q^2)_k}\left(\frac{1}{d}\right)^k$$
$$=\frac{(\omega^2q,-1;q^2)_\infty}{(-\omega^2qd,1/d;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-d,-\omega q,\omega/q;q^2)_k}{(-\omega qd,-1,q^2;q^2)_k}\left(\omega^2q\right)^k, \tag{3.17}$$

where the last equality comes from applying (3.3) to the series on the left.

Finally, after making the substitutions $c_1 = 1$, $a_1 = -q^3$ and $u = -q$ in the series on the left of (3.16) and the infinite product on the right side of

(3.16), let $d \to 0$[1], so that the sum over p on the left side of (3.16) is reduced to the single value $p = 0$. The sum on the left side of (3.16) becomes the sum on the right side of (3.15), after a relabelling of summation variables ($j_1 \to j$ and $k_1 \to k$). The left side of (3.16) simplifies to

$$(\omega^2 q, -1; q^2)_\infty \sum_{k=0}^{\infty} \frac{(-\omega q, \omega/q; q^2)_k}{(-1, q^2; q^2)_k} \left(\omega^2 q\right)^k = (-q^2; q^2)_\infty (\omega q, \omega^2 q; q^4)_\infty, \tag{3.18}$$

after applying (3.4) to the series. This is equivalent to the infinite product on the right side of (3.15). □

The next example is an illustration of Theorem 2.6. It also provides a third application of the identity at (3.6) (the first two being (3.9) and (3.10)). For readability reasons, the exponent of q on the left side of (3.19) has been slightly rearranged (writing $l(l-1)/2$ as $l^2/2 - l/2$, where l is as defined at (2.6)).

Example 6. *If* $|q| < 1$*, then*

$$\sum_{j,k,p,r=0}^{\infty} \frac{(-1)^{j+k} q^{(2j+k-p+r)^2/2+k(k+4)/2+3j-p/2+3r/2}(-q;q)_r}{\left(q^2;q^2\right)_j (q;q)_k (q;q)_p (q;q)_r} = 2\frac{\left(-q;q^2\right)_\infty \left(-q^2,-q^{14},q^{16};q^{16}\right)_\infty}{(q;q)_\infty}. \tag{3.19}$$

Proof. In (2.19), let $s = 2$ and $v = B = 1$. Set $m_1 = m_2 = 1$, $n_1 = 2$, $a_1 = -q^3$, $c_1 = -q^4$, $e_1 = q^3$, $f_1 = -q^2$, $d = -1/q$ and let $a_2 \to 0$, $u \to 0$ (so that the sums over i and k_2 vanish). After replacing j_1 with j, k_1 with k and r_1 with r, the left side of (2.19) becomes the left side of (3.19).

With the values stated above, the right side of (2.19) becomes

$$\frac{(q^2;q)_\infty(-1,-q;q)_\infty(-q^2;q)_\infty}{(-q^2;q^2)_\infty(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(-q^2;q^2)_k(q;q)_k}{(q^2;q)_k(q;q)_k(-q^2;q)_k}(-q)^k,$$

which simplifies to the right side of (3.19) upon using (3.6) (with the replacement $q \to -q$). □

Remark: We had initially believed the following identities to be new, but a reviewer of an earlier version of this paper pointed out that they all could be derived (and with less stringent conditions on the parameters than in the versions derived from the theorems in the present paper) simply from using only the q-binomial theorem.

$$\sum_{i,p=0}^{\infty} (-1)^{i+p} \frac{q^{(i-p)(i-p-1)/2} u^i d^p}{(q;q)_i (q;q)_p} = \frac{(u, qd; q)_\infty}{(ud;q)_\infty}. \tag{3.20}$$

[1]After substituting the right side of (3.17) for the left side in (3.16), and making the other indicated substitutions, the resulting identity is valid for $|qd| < 1$, by analytic continuation.

$$\sum_{i,j,k,p=0}^{\infty}(-1)^{i+j+p}\frac{q^{(i+j+k-p)(i+j+k-p-1)/2+k(k-1)/2}u^{i+k}c^{j+k}d^{p}}{(q;q)_i(q;q)_j(q;q)_k(q;q)_p} = \frac{(c,u,qd;q)_\infty}{(cd,ud;q)_\infty}. \quad (3.21)$$

$$\sum_{i,j,k,p=0}^{\infty}(-1)^{k+j+p}\frac{q^{(i+j+k-p)(i+j+k-p-1)/2+k(k-1)/2+i}c^{j+k}d^{p-i}}{(q;q)_i(q;q)_j(q;q)_k(q;q)_p} = \frac{(-q,qd;q)_\infty(c/d;q^2)_\infty}{(-qd;q)_\infty(cd;q^2)_\infty}. \quad (3.22)$$

The following identity, which follows from (2.18), may also be derived from repeated use of the q-binomial theorem together with some elementary finite q-product transformations.

$$\sum_{i,p,r=0}^{\infty}(-1)^{i+r+p}\frac{q^{(i+r-p)(i+r-p-1)/2}(u;q)_r u^i d^p f^r}{(q;q)_i(q;q)_p(q;q)_r} = \frac{(u,f,qd;q)_\infty}{(fd,ud;q)_\infty}. \quad (3.23)$$

4. Multi-sum Identities arising from Identities of Rogers–Ramanujan–Slater type

Almost all of the series-product identities on the Slater list of identities [24, 25] have a power of q that is *quadratic* in the exponent on the series side. On the other hand, all of the general multi-sum to single-sum transformations like those at (2.9), and elsewhere in the second section of this paper, have a term $(1/d)^k$ on the series side, so that any specialization of d as a negative power of q will result in a series in which the power of q is *linear* in the exponent.

For these reasons, some basic hypergeometric transformation has to be applied to the single series side of special cases of these general transformations before they can be used to derive multi-sum expansion for the infinite products in Slater-type identities.

In this section we consider one such transformation and give several applications. We start with Watson's transformation (see [10, page 360, III.18] or [19, page 33, Eq. (5.10)]), which states that for each positive integer n there holds

$$\sum_{k=0}^{n}\frac{1-aq^{2k}}{1-a}\frac{(a,b,c,h,e,q^{-n};q)_k}{(aq/b,aq/c,aq/h,aq/e,aq^{n+1},q;q)_k}\left(\frac{a^2q^{2+n}}{bche}\right)^k = \frac{(aq,aq/he;q)_n}{(aq/h,aq/e;q)_n}\sum_{j=0}^{n}\frac{(aq/bc,h,e,q^{-n};q)_j}{(aq/b,aq/c,heq^{-n}/a,q;q)_j}q^j. \quad (4.1)$$

Here we have replaced the d usually found in the statement of this transformation with h, to avoid confusion with the use of the parameter d in the various general multi-sum to single-sum transformations in section 2. Upon letting $n \to \infty$ and redefining a, b and c one arrives at the transformation

$$\sum_{k=0}^{\infty} \frac{(a,h,e;q)_k}{(f,g,q;q)_k} \left(\frac{fg}{ahe}\right)^k = \frac{(fg/ah, fg/ae;q)_\infty}{(fg/a, fg/ahe;q)_\infty} \times \sum_{k=0}^{\infty} \frac{1-fgq^{2k}/aq}{1-fg/aq} \frac{(fg/aq, f/a, g/a, h, e;q)_k}{(f,g,fg/ah, fg/ae, q;q)_k} \left(\frac{-fg}{he}\right)^k q^{k(k-1)/2}. \quad (4.2)$$

Next, after making the replacements $a \to ad, e \to de, h \to dh, f \to df, g \to dg$ followed by $e \to fg/(ah)$ one gets

$$\sum_{k=0}^{\infty} \frac{(ad,dh,de;q)_k}{\left(\frac{daeh}{g}, dg, q;q\right)_k} \left(\frac{1}{d}\right)^k = \frac{(h,e;q)_\infty}{\left(\frac{1}{d}, deh;q\right)_\infty} \sum_{k=0}^{\infty} \frac{\left(1-\frac{dehq^{2k}}{q}\right)}{\left(1-\frac{deh}{q}\right)} \frac{\left(\frac{eh}{g}, \frac{g}{a}, dh, de, \frac{deh}{q};q\right)_k}{\left(h, \frac{daeh}{g}, dg, e, q;q\right)_k} (-a)^k q^{k(k-1)/2}. \quad (4.3)$$

Next, the special case of (2.2) is constructed which has as the series on its right side the series on the left side of (4.3). This special case is achieved by setting $A = 2$, $a_1 = g$, $a_2 = aeh/g$, $c_1 = a$, $c_2 = e$ and $u = h$. Finally, (4.3) is used to replace the resulting series on the right side and the limit $d \to 0$ is taken (so that the sum over p on the left side of (2.2) vanishes) to get the identity in the next theorem (for readability, the replacements $j_1 \to j$, $k_1 \to k$, $k_2 \to l$ and $j_2 \to m$ are made).

Theorem 4.1. *Let*

$$Q(i,j,k,l,m) = \frac{(i+j+k+l+m)(i+j+k+l+m-1)}{2} + \frac{k(k-1)}{2} + \frac{l(l-1)}{2}.$$

If $|q| < 1$, *then*

$$\sum_{i,j,k,l,m=0}^{\infty} \frac{(-1)^{i+j+m} q^{Q(i,j,k,l,m)} a^j e^m g^k h^i (aeh/g)^l}{(q;q)_i (q;q)_j (q;q)_k (q;q)_l (q;q)_m} = (e,h;q)_\infty \sum_{k=0}^{\infty} \frac{(g/a, eh/g;q)_k}{(e,h,q;q)_k} (-a)^k q^{k(k-1)/2}. \quad (4.4)$$

For ease of use, we note two special cases.

Corollary 4.1. *If $|q| < 1$, then*

$$\sum_{i,k,m=0}^{\infty} \frac{(-1)^{i+m} q^{(i+k+m)(i+k+m-1)/2+k(k-1)/2} e^m g^k h^i}{(q;q)_i\,(q;q)_k (q;q)_m} = (e,h;q)_\infty \sum_{k=0}^{\infty} \frac{(eh/g;q)_k g^k q^{k(k-1)}}{(e,h,q;q)_k}, \quad (4.5)$$

$$\sum_{i,m=0}^{\infty} \frac{(-1)^{i+m} q^{(i+m)(i+m-1)/2} e^m h^i}{(q;q)_i (q;q)_m} = (e,h;q)_\infty \sum_{k=0}^{\infty} \frac{(-eh)^k q^{3k(k-1)/2}}{(e,h,q;q)_k}. \quad (4.6)$$

Proof. For (4.5), let $a \to 0$ in (4.4) (so that the sums over j and l on the left side vanish). Likewise, (4.6) follows upon letting $g \to 0$ in (4.5) (so that the sums over k on the left side vanishes). ☐

Before giving some applications (multi-sum representations of infinite products), we recall some identities from the Slater list (the labels refer to the identity number in [25], but the series side of the identities may be slightly re-written to fit the format of the transformations above, and the product side may have the formulation given by Sills [23]). We remark that there are additional identities on the Slater list [25] that may also be used to produce multi-sum identities similar to those in Example 7.

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(-q,-q^2,q^2;q^2)_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q^2;q^2)_\infty}. \quad \textbf{(S.19)}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^3,q^2;q^2)_n} = (1-q)\frac{(q^3,q^5,q^8;q^8)_\infty (q^2,q^{14};q^{16})_\infty}{(q;q)_\infty}. \quad \textbf{(S.38)}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q,q^2;q^2)_n} = \frac{(q,q^7,q^8;q^8)_\infty (q^6,q^{10};q^{16})_\infty}{(q;q)_\infty}. \quad \textbf{(S.39)}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(-q^{1/2},q^{1/2},q;q)_n} = \frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q;q)_\infty}. \quad \textbf{(S.46)}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(-q^{1/2},q^{1/2},q;q)_n} = 1 + 2q\frac{(-q,-q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty}. \quad \textbf{(S.56)}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(3n+2)}}{(q^3,-q^2,q^2;q^2)_n} = (1-q)\frac{(q^3,q^7,q^{10};q^{10})_\infty (q^4,q^{16};q^{20})_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty}. \quad \textbf{(S.97)}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q,-q^2,q^2;q^2)_n} = \frac{(q,q^9,q^{10};q^{10})_\infty (q^8,q^{12};q^{20})_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty}. \quad \textbf{(S.100)}$$

We recall one additional identity ([17, page 15, Eq. (2.8.11)]):

$$\sum_{n=0}^{\infty} \frac{(-q^{-1}, -q^5; q^4)_n q^{2n^2}}{(q^2, -q^4, q^4; q^4)_n} = \frac{(-q, -q^7, q^8; q^8)_\infty (-q^2; q^4)_\infty}{(q^4; q^4)_\infty}. \tag{4.7}$$

We now use the above identities together with the transformations (4.4) - (4.6) to derive multi-sum expansions for some infinite products.

Remark: The identities given in the next example possibly may be regarded as being not true identities of Kanade–Russell type, since the finite q-products in the denominator of the general term on the series sides all have the same modulus. The series sides can probably be more correctly classified as being "Andrews–Gordon type series" (see (1.15)).

Example 7. *If $|q| < 1$, then the following identities hold.*

$$\sum_{i,m=0}^{\infty} \frac{q^{(i+m)^2+i}}{(q^2; q^2)_i (q^2; q^2)_m} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q; q)_\infty}. \tag{4.8}$$

$$\sum_{i,k=0}^{\infty} \frac{(-1)^i q^{(i+k+1)^2+k^2-1}}{(q^2; q^2)_i (q^2; q^2)_k} = \frac{(q^3, q^5, q^8; q^8)_\infty (q^2, q^{14}; q^{16})_\infty}{(q^2; q^2)_\infty}. \tag{4.9}$$

$$\sum_{i,k=0}^{\infty} \frac{(-1)^i q^{(i+k)^2+k^2}}{(q^2; q^2)_i (q^2; q^2)_k} = \frac{(q, q^7, q^8; q^8)_\infty (q^6, q^{10}; q^{16})_\infty}{(q^2; q^2)_\infty}. \tag{4.10}$$

$$\sum_{i,m=0}^{\infty} \frac{(-1)^i q^{(i+m)^2/2}}{(q; q)_i (q; q)_m} = \frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q^2; q^2)_\infty}. \tag{4.11}$$

$$\sum_{i,k,m=0}^{\infty} \frac{(-1)^i q^{((i+k+m)^2+k^2)/2}}{(q; q)_i (q; q)_k (q; q)_m} = (q; q^2)_\infty + 2q \frac{(-q, -q^{11}, q^{12}; q^{12})_\infty}{(q^2; q^2)_\infty}. \tag{4.12}$$

$$\sum_{i,m=0}^{\infty} \frac{(-1)^i q^{(i+m)(i+m+1)+i}}{(q^2; q^2)_i (q^2; q^2)_m} = \frac{(q^3, q^7, q^{10}; q^{10})_\infty (q^4, q^{16}; q^{20})_\infty}{(q^2; q^2)_\infty}. \tag{4.13}$$

$$\sum_{i,m=0}^{\infty} \frac{(-1)^i q^{(i+m)^2+m}}{(q^2; q^2)_i (q^2; q^2)_m} = \frac{(q, q^9, q^{10}; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}{(q^2; q^2)_\infty}. \tag{4.14}$$

$$\sum_{i,j,k,l,m=0}^{\infty} \frac{(-1)^i q^{2(i+k+j+l+m)^2+l(2l+3)+k(2k-3)+2m}}{(q^4; q^4)_i (q^4; q^4)_j (q^4; q^4)_k (q^4; q^4)_l (q^4; q^4)_m} = \frac{(-q, -q^7, q^8; q^8)_\infty}{(q^4; q^4)_\infty}. \tag{4.15}$$

Proof. Remark: In all the identities stated above, the exponent in the power of q on the series side has been simplified.

For (4.8), replace q with q^2 in (4.6), then set $e = -q$, $h = -q^2$, and use (**S.19**). Note that (4.8) may also be proved by using the change in summation variable $N = i + j$ on the left side, then employing the identity

$$\sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_{q^2} q^i = (-q;q)_N,$$

and finally using one of the Rogers–Ramanujan identities.

In (4.5), replace q with q^2 and set $g = q^4$, $e = 0$ and $h = q^3$. After employing (**S.38**) then (4.9) follows. Here also (4.9) may also be proved by using a change in summation variable $N = i + k - 1$ on the left side, then employing a special case of the q-binomial theorem,

$$\sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_{q^2} (-1)^i q^{i^2} = (q;q)_N,$$

and finally using another identity in the Slater list ([25, page 155, Eq. (34)], after replacing q with $-q$):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n} = \frac{(q,q^7,q^8;q^8)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

For (4.10), replace q with q^2 in (4.5) and then set $g = q^2$, $e = 0$ and $h = q$, and finally use (**S.39**). Note that (4.10) has an alternative proof similar to the alternative proofs of (4.8) and (4.9). However, we omit the details, and while similar alternative proofs may exist for the remaining identities proved below, we do not consider that possibility further.

In (4.6), set $e = \sqrt{q}$, $h = -\sqrt{q}$, then use (**S.46**), and (4.11) follows.

Identity (4.12) follows from (4.5), upon making the replacements $e = \sqrt{q}$, $h = -\sqrt{q}$, $g = q$ and then using (**S.56**).

For (4.13), replace q with q^2 in (4.6), set $e = -q^2$, $h = q^3$, and then use (**S.97**). Identity (4.14) follows similarly from replacing q with q^2 in (4.6), this time setting $e = -q^2$ and $h = q$, and then using (**S.100**).

Finally, for (4.15) replace q with q^4 in (4.6), set $a = -q^2$, $e = q^2$, $g = q^7$, $h = -q^4$ and then use (4.7).

□

Remarks: (1) The identity (4.14) was also proven by Sills [23, page 79, Eq. (5.17)], using different methods.

(2) One of the anonymous referees asked if a number of similar identities, which had been verified by them experimentally, could be proved by the methods of the paper:

$$\sum_{i,m=0}^{\infty} \frac{q^{(i+m)^2+2i+m}}{(q^2;q^2)_i(q^2;q^2)_m} = \frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}. \tag{4.16}$$

$$\sum_{i,k=0}^{\infty} \frac{(-1)^i q^{(i+k)(i+k+2)}}{(q^2;q^2)_i(q^2;q^2)_k} = \frac{1}{(q^8, q^{12}; q^{20})_\infty}. \tag{4.17}$$

The identity (4.16) follows from (4.6) upon replacing q with q^2, setting $e = -q^3$, $f = -q^2$ and then employing an identity of Ramanujan (see [5, p. 252 (11.2.7)], or [17, p.12 (2.5.6)])

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+2n}}{(-q;q^2)_{n+1}(q^4;q^4)_n} = \frac{(q, q^4, q^5; q^5)_\infty}{(q^2;q^2)_\infty}. \tag{4.18}$$

to sum the resulting right side. Likewise (4.17) follows from (4.6) upon replacing q with q^2, setting $e = -q^3$, $f = q^3$ and using an identity of Rogers ([21, p. 330 (2), line 2])

$$\sum_{n=0}^{\infty} \frac{q^{3(n^2+n)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(q^2, q^8, q^{10}; q^{10})_\infty}{(q;q)_\infty}. \tag{4.19}$$

(with q replaced with q^2) to sum the resulting right side.

(3) The products at (4.9) and (4.10) are related (but not equal) to products appearing in principal characters of level 5 modules for $A_2^{(2)}$ in the recent paper [14] of Kanade and Russell. Likewise, the products at (4.11) (with q replaced by q^2), (4.13), (4.14) and (4.17) are similar to products related to level 7 modules for $A_2^{(2)}$ in the same paper.

5. Concluding Remarks

There is an obvious obstruction to the various theorems having more widespread application to producing multi-sum-to-infinite product identities from identities of Rogers–Ramanujan–Slater type, similar to those in the papers by Andrews [2] and Sills [23]. As mentioned at the start of section 4, this obstruction stems from the fact that the exponent in $(1/d)^k$ factor in the single series on the right side of each of the identities in the theorems is clearly linear in k rather than quadratic. Further, it is not possible to produce such a term that is quadratic in the exponent without applying some transformation to the series, as was done at (4.4) above. Other transformations may lead to similar applications.

Another limitation is the existence of $-p$ in the formula for l in each of the general transformations, for example

$$l = i - p + j_1 + \cdots + j_A + k_1 + \cdots + k_A$$

in Theorem 2.2. This means that the theorems stated here cannot be used to directly prove any of the summations stated in the introduction where the variables in the exponent of q on the series side are all positive, the proof of any of these requiring a transformation of the single series that would allow $d \to 0$ (as was done at (3.17) in the proof of one of the Kanade–Russell identities, and at (4.4)).

However, we believe that the usefulness of the general transformations described in the present paper has been demonstrated. The reader may be able to find additional applications. We also leave to the reader the possibility of finding combinatorial interpretations of some of the identities stated in the present paper and/or finding combinatorial proofs of them.

6. Conflict of Interest Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Andrews, G. E. *An analytic generalization of the Rogers–Ramanujan identities for odd moduli.* Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 4082–4085.
[2] G. E. Andrews, Multiple series Rogers–Ramanujan type identities. Pacific J. Math. 114 (1984) 267–283.
[3] Andrews, G. E. *Schur's theorem, Capparelli's conjecture and q-trinomial coefficients*, in The Rademacher Legacy to Mathematics, Amer. Math. Soc., 1994, pp. 141–154.
[4] Andrews, G. E. *The theory of partitions.* Reprint of the 1976 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. xvi+255 pp.
[5] G.E. Andrews and B.C. Berndt, *Ramanujan's Lost Notebook, Part I*, Springer, 2005.
[6] Berkovich, A.; Uncu, A. K. *Elementary polynomial identities involving q-trinomial coefficients.* Ann. Comb. **23** (2019), no. 3–4, 549–560.
[7] Bringmann, K.; Jennings–Shaffer, C.; Mahlburg, K. *Proofs and reductions of various conjectured partition identities of Kanade and Russell.* J. Reine Angew. Math. **766** (2020), 109–135.
[8] Capparelli, S. *Vertex operator relations for affine algebras and combinatorial identities*, Ph.D. Thesis, Rutgers University, 1988.
[9] Capparelli, S. *A combinatorial proof of a partition identity related to the level 3 representations of a twisted affine Lie algebra.* Comm. Algebra **23** (1995), no. 8, 2959–2969
[10] Gasper, G.; Rahman, M. *Basic hypergeometric series.* With a foreword by Richard Askey. Second edition. Encyclopedia of Mathematics and its Applications, 96. Cambridge University Press, Cambridge, 2004. xxvi+428 pp.
[11] F. H. Jackson, Transformation of q-series, Messenger of Math. 39 (1910) 145–153.
[12] Kanade, S., Russell, M.C. *IdentityFinder and some new identities of Rogers–Ramanujan type*, Experimental Mathematics **24** (2015), 419–423.
[13] Kanade, S., Russell, M.C. *Staircases to analytic sum-sides for many new integer partition identities of Rogers–Ramanujan type.* Electron. J. Combin. **26**, 1–6 (2019).
[14] Kanade, S., Russell, M. C. *On q-series for principal characters of standard $A_2^{(2)}$-modules.* Adv. Math. **400** (2022), Paper No. 108282.
[15] Kurşungöz, K. *Andrews–Gordon type series for Capparelli's and Göllnitz–Gordon identities.* J. Combin. Theory Ser. A **165** (2019), 117–138.
[16] Kurşungöz, K. *Andrews–Gordon type series for Kanade–Russell conjectures.* Ann. Comb. **23** (2019), no. 3–4, 835–888.
[17] Mc Laughlin, J.; Sills, A. V.; Zimmer, P. *Rogers–Ramanujan–Slater Type Identities* Electronic Journal of Combinatorics **15** (2008) #DS15, 59 pp.
[18] Mc Laughlin, J.; Sills, A. V.; Zimmer, P. *Some implications of Chu's ${}_{10}\psi_{10}$ extension of Bailey's ${}_6\psi_6$ summation formula* Online J. Anal. Comb. No. **5** (2010), 24 pp.
[19] Mc Laughlin, J. *Topics and methods in q-series.* With a foreword by George E. Andrews. Monographs in Number Theory, **8**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018. x+390 pp.

[20] Nandi, D. *Partition identities arising from the standard $A_2^{(2)}$ -modules of level 4*, Ph.D. Thesis, Rutgers University, 2014.

[21] Rogers L. J. (1917), On two theorems of combinatory analysis and some allied identities, *Proc. London Math. Soc (2)*. **16**, 315–336.

[22] Rosengren, H. *Proofs of some partition identities conjectured by Kanade and Russell.* Ramanujan J (online, 2021). https://doi.org/10.1007/s11139-021-00389-9

[23] A. V. Sills, *Finite Rogers–Ramanujan type identities.* Electronic J. Combin. **10(1)** (2003) #R13, 1–122.

[24] Slater, L. J. *A new proof of Rogers's transformations of infinite series.* Proc. London Math. Soc. (2) 53, (1951). 460–475.

[25] Slater, L. J. *Further identities of the Rogers–Ramanujan type*, Proc. London Math.Soc. **54** (1952) 147–167.

[26] Takigiku, M.; Tsuchioka, S. *Andrews–Gordon type series for the level 5 and 7 standard modules of the affine Lie algebra $A_2^{(2)}$.* Proc. Amer. Math. Soc. **149** (2021), no. 7, 2763–2776.

[27] Takigiku, M.; Tsuchioka, S. *A proof of conjectured partition identities of Nandi*, arXiv:1910.12461

[28] Tamba, M.; Xie, C. F. *Level three standard modules for $A_2^{(2)}$ and combinatorial identities.* J. Pure Appl. Algebra **105** (1995), no. 1, 53–92.

Mathematics Department, 25 University Avenue, West Chester University, West Chester, PA 19383

E-mail address: jmclaughlin2@wcupa.edu