SOME MORE IDENTITIES OF THE ROGERS-RAMANUJAN TYPE

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ABSTRACT. In this we paper we prove several new identities of the Rogers-Ramanujan-Slater type. These identities were found as the result of computer searches. The proofs involve a variety of techniques, including series-series identities, Bailey pairs, a theorem of Watson on basic hypergeometric series, generating functions and miscellaneous methods.

1. INTRODUCTION

The most famous of the "q-series=product" identities are the Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}. \tag{1.2}$$

These identities have a curious history ([15], p. 28). They were first proved by L.J. Rogers in 1894 ([20]) in a paper that was completely ignored. They were rediscovered (without proof) by Ramanujan sometime before 1913. In 1917, Ramanujan rediscovered Rogers's paper. Also in 1917, these identities were rediscovered and proved independently by Issai Schur ([22]). They were also discovered independently by R. Baxter (see [4] for details). An account of the many proofs of the Rogers-Ramanujan identities can be found in [3].

There are numerous identities that are similar to the Rogers-Ramanujan identities. These include identities by Jackson ([16]), Rogers ([20] and [21]) and Bailey ([8] and [9]). Of special note is Slater's 1952 paper [27], which contains a list of 130 such identities, many of them new (see the paper by the third author [24], for an annotated version of Slater's list). There are also other identities of Rogers-Ramanujan type in the literature.

In the present paper we describe the results of some numerical investigations, which were undertaken with the aim of finding new Rogers-Ramanujan type identities. One reason for searching for such identities is the possibility that new identities might be found experimentally which could *not* be proven using presently existing methods, thus necessitating developments in the mathematical theory.

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These investigations did uncover several new identities, although all were provable using known methods. The proof of these identities lets us present examples of the various methods used to prove identities of the Rogers-Ramanujan type. These methods include using series-series identities, Bailey pairs, generating functions and some miscellaneous methods.

One search involved computing to high precision series of the form

$$S := \sum_{n=0}^{\infty} \frac{q^{(an^2+bn)/2}(-1)^{cn}(d,e;q)_n}{(f,g,q;q)_n}, \tag{1.3}$$

for a fixed numerical value of q, for

$$d,e,f,g \in \{0,-1,q,-q,-q^2,q^2\}, \qquad c \in \{0,1\},$$

and for integers a and b. This choice for the form of S was motivated by the fact that many series on the Slater list have this form.

For each particular choice of the parameters a, b, c, d, e, f and g, a numerical comparison was performed to see if

$$S - \prod_{j=1}^{L}(q^j;q^L)_\infty^{s_j} = 0, \tag{1.4}$$

for integers s_j and $L \in \{20,24,28,32,36\}$. With sufficient precision, a small numerical value for the left side of (1.4) indicated either a known identity or a potential new identity, which then needed to be proved.

We also tried searches where the series had the forms

$$S' := \sum_{n=0}^{\infty} \frac{q^{(an^2+bn)/2}(-1)^{cn}(d,e;q^2)_n}{(f,g,q^2;q^2)_n},$$
$$S'' := \sum_{n=0}^{\infty} \frac{q^{(an^2+bn)/2}(-1)^{cn}(d;q)_n}{(e;q^2)_{n+1}(q;q)_{n+1}},$$

the latter form being motivated by identity (56) on Slater's list.

It is possible that other choices for the form of the series, or extending the choices for the various parameters, may turn up new identities. The computations associated with the searches described above were performed using *PARI/GP*.

2. Some Identities Discovered using PARI/GP

In this section we list the new identities found during the *PARI/GP* searches. We organize them according to the methods used in their proofs.

2.1. Infinite Series Transformations.

Infinite Series Transformations. Infinite series transformations can be used to derive new identities from known identities, since if one of the series can be expressed as an infinite product, for certain values of the parameters, then the other series automatically also has an expression as an infinite product. Before coming to the identities in this subsection, we list some infinite series transformations that are necessary for the proofs.

We first recall Heine's q-Gauss sum.

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n}{(c,q;q)_n}\left(\frac{c}{ab}\right)^n = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}$$

Let $b \to \infty$ to get

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}(-c/a)^n}{(c;q)_n(q;q)_n} = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{2.1}$$

The second identity we need is the following.

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(b;q)_n(q;q)_n} = \frac{(-\gamma;q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(-a\gamma/b;q)_n q^{n(n-1)/2}(-b)^n}{(-\gamma;q)_n(q;q)_n}, \tag{2.2}$$

This identity, in a more symmetric form, is found in Ramanujan's lost notebook [19] and a proof can be found in the recent book by Andrews and Berndt [5]. It also follows from the second iteration of Heine's transformation for a ${}_2\phi_1$ series [14, equations III.1 and III.2, page 359]. The form at (2.2) is better suited to our present requirements than Ramanujan's more symmetric form.

We also recall the following transformation [14, page 80]:

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{(1-aq^{4n})(a,b,c,cq,d,dq;q^2)_n}{(1-a)(aq^2/b,aq^2/c,aq/c,aq^2/d,aq/d,q^2;q^2)_n}\left(\frac{a^2q^2}{bc^2d^2}\right)^n \\ &= \frac{(aq,aq/bc;q)_\infty(a^2q^2/c^2d^2,a^2q^2/d^2b;q^2)_\infty}{(aq/b,aq/c;q)_\infty(a^2q^2/d^2,a^2q^2/c^2d^2b;q^2)_\infty} \\ &\quad\times \sum_{n=0}^{\infty} \frac{(1+a/dq^{2n})(-a/d,c;q)_n(b,aq/d^2;q^2)_n}{(1+a/d)(-aq/cd,q;q)n(a^2q^2/d^2b,aq;q^2)_n}\left(\frac{aq}{bc}\right)^n.\end{aligned}$$

Upon replacing c with aq/c, then letting $a \to 0$ and finally $b \to \infty$, we get

$$\sum_{n=0}^{\infty} \frac{(d;q)_{2n} q^{n^2-n}(-c^2/d^2)^n}{(q^2;q^2)_n(c;q)_{2n}} = \frac{(c^2/d^2;q^2)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-c)^n}{(q;q)_n(-c/d;q)_n}. \tag{2.3}$$

Theorem 2.1.

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}. \tag{2.4}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q)_{2n} q^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_n} = \frac{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.5}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{2n^2+4n}(-1)^n}{(q^8;q^8)_n(-q^2;q^4)_{n+1}} = (-q^9,-q^7,q^8;q^8)_\infty \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{2.6}$$

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{2n^2}(-1)^n}{(q^8;q^8)_n(-q^2;q^4)_n} = (-q^3,-q^5,q^8;q^8)_\infty \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{2.7}$$

Proof. For (2.4), replace q by q^2 in (2.1), and set $a = -q^2$ and $c = q^3$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} &= \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q^3;q^2)_n(q^2;q^2)_n} \\ &= \frac{1}{1-q} \frac{(-q;q^2)_\infty}{(q^3;q^2)_\infty} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.\end{aligned}$$

To prove (2.5), we use the following identity ((25) from Slater's list, with q replaced by $-q$):

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q^3;q^6)_\infty^2 (q^6;q^6)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.8}$$

In (2.2), replace q by q^2 and set $a=-1$, $b=q$ and $\gamma=q^2$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1;q)_{2n} q^{n^2+n}}{(q^2;q^2)_n (q^2;q^4)_n} \\
&= \sum_{n=0}^{\infty} \frac{(-1;q^2)_n (-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n (q;q^2)_n (-q;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2-n} (q^2)^n}{(q;q^2)_n (q^2;q^2)_n} \\
&= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q)^n q^{n^2-n} (q;q^2)_n}{(-q^2;q^2)_n (q^2;q^2)_n} \\
&= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q^2;q^2)_n (q^2;q^2)_n} \\
&= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(q^4;q^4)_n} \\
&= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \frac{(-q^3;q^6)_\infty^2 (q^6;q^6)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty} \quad \text{(by (2.8))}.
\end{aligned}$$

The result at (2.5) now follows.

Before proving (2.6) and (2.6), we recall two identities from [27] (identities (38) and (39) (the latter was also stated by Jackson [16]):

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{(q^3,q^5,q^8;q^8)_\infty (q^2,q^{14};q^{16})_\infty}{(q;q)_\infty}, \tag{2.9}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{(q,q^7,q^8;q^8)_\infty (q^6,q^{10};q^{16})_\infty}{(q;q)_\infty}. \tag{2.10}$$

To prove (2.6), we replace q by q^2 in (2.3) and then let $c=-q^4$ and $d=q$ to get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{2n^2+4n} (-1)^n}{(q^4;q^4)_n (-q^4;q^2)_{2n}} = \frac{(q^6;q^4)_\infty}{(-q^4;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n (q^3;q^2)_n} \\
\Longrightarrow &\sum_{n=0}^{\infty} \frac{(q;q^2)_{2n} q^{2n^2+4n} (-1)^n}{(q^8;q^8)_n (-q^2;q^4)_{n+1}} = \frac{(q^2;q^4)_\infty}{(1+q)(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q)_{2n+1}} \\
&\qquad = \frac{(q^2;q^4)_\infty}{(1+q)(-q^2;q^2)_\infty} \frac{(q^3,q^5,q^8;q^8)_\infty (q^2,q^{14};q^{16})_\infty}{(q;q)_\infty},
\end{aligned}$$

where the last equality follows from (2.9). The result now follows, after some elementary infinite product manipulations. The proof of (2.7) is similar, except we set $c=-q^2$, $d=q$ and use (2.10). □

Remark: Shortly after proving (2.4), we discovered, while reading a pre-print version of [6], that it had previously been stated by Ramanujan (see [6, **Entry 1.7.13**]). However, we include since we discovered it independently and [6] has not yet been published.

2.2. **Watson's Transformation.** Before proving the next identity, we introduce some notation.

An ${}_r\phi_s$ *basic hypergeometric series* is defined by

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(q;q)_n (b_1;q)_n \dots (b_s;q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{s+1-r} x^n, $$

for $|q| < 1$.

Watson's identity is the following.

$$ (2.11)\quad {}_8\phi_7\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-n} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right) = \frac{(aq)_n (aq/de)_n}{(aq/d)_n (aq/e)_n}\ {}_4\phi_3\left(\begin{matrix} aq/bc, d, e, q^{-n} \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q\right), $$

where n is a non-negative integer. Watson [29] used his transformation in his proof of the Rogers-Ramanujan identities (1.1).

Theorem 2.2. *Let a, b and $q \in \mathbb{C}$, with $|q| < 1$. Then*

$$ (2.12)\quad \sum_{r=0}^{\infty} \frac{(1+aq^r)(a^2;q)_r (b;q)_r \,(-a/b)^r\, q^{r(r+1)/2}}{(a^2q/b;q)_r (q;q)_r} = \frac{(-a;q)_\infty (a^2q;q^2)_\infty (aq/b;q)_\infty}{(a^2q/b;q)_\infty}. $$

Proof. We will use Bailey's identity [7]:

$$ (2.13)\quad \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(aq/b;q)_n (q;q)_n} \left(-\frac{q}{b}\right)^n = \frac{(aq;q^2)_\infty (-q;q)_\infty (aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty (-q/b;q)_\infty}. $$

First, let $n, b \to \infty$ in (2.11) to get

$$ (2.14)\quad \sum_{r\geq 0} \frac{(1-aq^{2r})(a)_r (c)_r (d)_r (e)_r \left(a^2/cde\right)^r q^{r(r-1)+2r}}{(1-a)(aq/c)_r (aq/d)_r (aq/e)_r (q)_r} = \frac{(aq)_\infty (aq/de)_\infty}{(aq/d)_\infty (aq/e)_\infty} \sum_{r\geq 0} \frac{(d)_r (e)_r (aq/de)^r}{(aq/c)_r (q)_r}. $$

Next, replace a by $-a$, set $c = -b$, $d = a$ and $e = b$, so that (2.14) becomes

$$ \sum_{r\geq 0} \frac{(1+aq^{2r})(-a)_r (-b)_r (a)_r (b)_r \left(-a/b^2\right)^r q^{r(r-1)+2r}}{(1+a)(aq/b)_r (-q)_r (-aq/b)_r (q)_r} = \frac{(-aq)_\infty (-q/b)_\infty}{(-q)_\infty (-aq/b)_\infty} \sum_{r\geq 0} \frac{(a)_r (b)_r (-q/b)^r}{(aq/b)_r (q)_r}, $$

and the result now follows from (2.13), after replacing b^2 by b and q^2 by q. Note that the result holds initially for $|q/b| < 1$, and then follows for general b by analytic continuation. □

2.3. **Bailey Pairs.** A pair of sequences (α_n, β_n) that satisfy $\alpha_0 = 1$ and

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{2.15}$$

is termed a *Bailey pair relative to a*. Bailey [8, 9] showed that, for such a pair,

$$\sum_{n=0}^{\infty} (y,z;q)_n \left(\frac{aq}{yz}\right)^n \beta_n = \frac{(aq/y, aq/z;q)_\infty}{(aq, aq/yz;q)_\infty} \sum_{n=0}^{\infty} \frac{(y,z;q)_n}{(aq/y, aq/z;q)_n} \left(\frac{aq}{yz}\right)^n \alpha_n. \tag{2.16}$$

We note two special cases which will be needed later. Firstly, upon letting y, $z \to \infty$ we get that

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n. \tag{2.17}$$

Secondly, upon setting $y = q^{1/2}$ and letting $z \to \infty$ we get that

$$\sum_{n=0}^{\infty} (q^{1/2};q)_n (-1)^n a^n q^{n^2/2} \beta_n = \frac{(aq^{1/2};q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{1/2};q)_n}{(aq^{1/2};q)_n} a^n (-1)^n q^{n^2/2} \alpha_n. \tag{2.18}$$

Lemma 2.3. *The pair (α_n, β_n) is a Bailey pair relative to 1, where*

$$\alpha_n = \begin{cases} 1, & n = 0, \\ 2(-1)^n q^{n^2/2}, & n \geq 1, \end{cases}$$

$$\beta_n = \frac{(\sqrt{q};q)_n}{(-\sqrt{q}, -q, q;q)_n}.$$

Proof. Set $a = 1$, $c = -\sqrt{q}$, $d = -1$ in Slater's equation (4.1) from [26, page 468]:

$$\sum_{r=0}^{n} \frac{(1-aq^{2r})(a,c,d;q)_r q^{(r^2+r)/2}}{(a;q)_{n+r+1}(q;q)_{n-r}(aq/c, aq/d, q;q)_r} \left(\frac{-a}{cd}\right)^r = \frac{(aq/cd;q)_n}{(aq/c, aq/d, q;q)_n}.$$

The result follows from (2.15), after a little simplification. □

Theorem 2.4.

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n}{(-q;q^2)_n (q^4;q^4)_n} = \frac{(q^3;q^6)_\infty^2 (q^6;q^6)_\infty}{(q^2;q^2)_\infty}. \tag{2.19}$$

Proof. Substitute the Bailey pair from Lemma 2.3 into (2.17), with $a = 1$, and replace q with q^2. The result follows after using using the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-q/z, -qz, q^2;q^2)_\infty \tag{2.20}$$

to sum the resulting right side. □

Remark: This is a companion identity to number (27) on Slater's list, with q replaced by $-q$:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty}.$$

2.4. **An identity of Bailey.** Before coming to the next identity we recall a result of Bailey([10], p. 220):

$$(2.21)\quad (-z^2q,-z^{-2}q^3,q^4;q^4)_\infty + z(-z^2q^3,-z^{-2}q,q^4;q^4)_\infty = (-z,-z^{-1}q,q;q)_\infty.$$

We also recall Slater's Bailey pair **G3** (relative to 1) from [26].

$$(2.22)\qquad \alpha_n = \begin{cases} 1, & n=0,\\ q^{3r^2}(q^{3r/2}+q^{-3r/2}), & n=2r,\ r\ge 1,\\ -q^{3r^2}(q^{3r/2}+q^{9r/2+3/2}), & n=2r+1, \end{cases}$$

$$\beta_n = \frac{q^n}{(q^2;q^2)_n(-q^{1/2};q)_n}.$$

We note that Slater used **G3** to derive two other series-product identities, (16) and (32) in [27], so we may regard the identity in Theorem 2.5 as one she missed. [1]

Theorem 2.5. *Let* $|q|<1$. *Then*

$$(2.23)\qquad \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Remark: This identity is clearly a companion to Identity (21) on Slater's list:

$$(2.24)\qquad \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3;q^5)_\infty(-q^2;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Proof of Theorem 2.5. We insert the Bailey pair (2.22) into (2.18), set $a=1$ and replace q by q^2 to get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n}\\
&\quad= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{r=1}^{\infty} q^{10r^2}(q^{3r}+q^{-3r}) + \sum_{r=0}^{\infty} q^{10r^2+4r+1}(q^{3r}+q^{9r+3})\right)\\
&\quad= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{r=-\infty}^{\infty} q^{10r^2+3r} + q^4\sum_{r=-\infty}^{\infty} q^{10r^2+13r}\right)\\
&\quad= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left((-q^7,-q^{13},q^{20};q^{20})_\infty + q^4(-q^{-3},-q^{23},q^{20};q^{20})_\infty\right)\\
&\quad= \frac{(-q^3;q^5)_\infty(-q^2;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}.
\end{aligned}$$

The next-to-last equation follows from (2.20), and the last equation follows from (2.21), with q replaced by q^5 and $z=q^4$. □

[1] In an earlier version of this paper we proved Theorem 2.5 by the method of q-difference equations. However, that proof was much longer and less transparent than the present proof.

2.5. **Miscellaneous Methods.** Before coming to the next identity, we recall two other necessary results. The first of these is an identity of Blecksmith, Brillhart and Gerst [11] (a proof is also given in [13]):

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q\frac{(q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20}; q^{20})_\infty}{(q^3, q^7, q^8, q^{12}, q^{13}, q^{17}; q^{20})_\infty}. \tag{2.25}$$

If we replace q by $-q$ and apply the Jacobi triple product identity to the left side, (2.25) may be re-written as

$$(q^5, q^5, q^{10}; q^{10})_\infty - (q, q, q^2; q^2)_\infty = 2q\frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(-q^3, -q^7; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}. \tag{2.26}$$

The second is the following identity, due to Rogers [21, p. 330 (4), line 3, corrected] recently generalized by the third author [25, p. 404, Eq. (3)]:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1; q)_n}{(q; q)_n (q; q^2)_n} = \frac{(q^5, q^5, q^{10}; q^{10})_\infty}{(q; q)_\infty (q; q^2)_\infty}. \tag{2.27}$$

We are now able to prove another identity discovered during the present investigations.

Theorem 2.6. *Let $|q| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q; q)_n}{(q; q^2)_{n+1} (q; q)_{n+1}} = \frac{(q^{10}; q^{10})_\infty}{(q; q)_\infty (q; q^2)_\infty (-q^3, -q^4, -q^6, -q^7; q^{10})_\infty}. \tag{2.28}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q; q)_n}{(q; q^2)_{n+1} (q; q)_{n+1}} &= \frac{1}{2q} \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-1; q)_{n+1}}{(q; q^2)_{n+1} (q; q)_{n+1}} \\
&= \frac{1}{2q} \left(\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1; q)_n}{(q; q^2)_n (q; q)_n} - 1 \right) \\
&= \frac{1}{2q} \left(\frac{(q^5, q^5, q^{10}; q^{10})_\infty}{(q; q)_\infty (q; q^2)_\infty} - 1 \right) \text{ (by (2.27))} \\
&= \frac{1}{2q (q; q)_\infty (q; q^2)_\infty} \left((q^5, q^5, q^{10}; q^{10})_\infty - (q; q^2)_\infty (q; q^2)_\infty (q^2; q^2)_\infty \right) \\
&= \frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q; q)_\infty (q; q^2)_\infty (-q^3, -q^7; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty} \text{ (by (2.26))} \\
&= \frac{(q^{10}; q^{10})_\infty}{(q; q)_\infty (q; q^2)_\infty (-q^3, -q^4, -q^6, -q^7; q^{10})_\infty}.
\end{aligned}$$

□

3. An Application of the Two-Variable Generalization of a Rogers-Ramanujan Type Series

In [2], Andrews showed that a certain two-variable generalization $f(t, q)$ of a Rogers-Ramanujan type series $\Sigma(q)$ served as a generating function in t of a sequence of polynomials $P_n(q)$ for which $\lim_{n\to\infty} P_n(q) = \Sigma(q)$.

For example, if

$$\Sigma := \Sigma(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}, \tag{3.1}$$

the series associated with the first Rogers-Ramanujan identity, and the two variable generalization $f(t,q)$ of Σ is given by

$$f(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(t;q)_{n+1}}, \tag{3.2}$$

then it is also the case that

$$f(t,q) = \sum_{n=0}^{\infty} P_n(q) t^n \tag{3.3}$$

where

$$P_0(q) = P_1(q) = 1; \qquad P_n(q) = P_{n-1}(q) + q^{n-1} P_{n-2}(q) \quad \text{if } n \geq 2. \tag{3.4}$$

Note that the polynomials in (3.4), which are q-analogs of the Fibonacci numbers, are sometimes called the *Schur polynomials* because they were employed by Schur in his proof of the Rogers-Ramanujan identities. Indeed, Schur [22] showed that

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n+5j+1}{2} \rfloor \end{bmatrix}_q, \tag{3.5}$$

while elsewhere MacMahon [17] showed that

$$P_n(q) = \sum_{j \geq 0} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q, \tag{3.6}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_q := \begin{cases} (q;q)_A (q;q)_B^{-1} (q;q)_{A-B}^{-1} & \text{if } 0 \leq B \leq A \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by combining (3.6) and (3.5), we may observe, as Andrews did in [1], that we have a polynomial identity which generalizes the first Rogers-Ramanujan identity and that we may recover (1.1) by letting $n \to \infty$.

In [24], the third author used these two-variable generalizations $f(t,q)$ of Rogers-Ramanujan type series to find polynomial generalizations of all 130 identities in Slater's list [27]. Previously, Santos [23] had studied a large number of polynomial sequences associated with two-variable generalizations of series in Slater's list. Indeed, the primary use of the $f(t,q)$ has been as a generating function in t for sequences of polynomials.

However, here we wish to turn our attention to a different use of the $f(t,q)$ by following up on an observation made by Andrews [2, p. 89]. Letting $f(q,t)$ and $\Sigma(q)$ be as above, Andrews noted the following: not only do we have, as required,

$$\lim_{t \to 1^-} (1-t) f(t,q) = \Sigma(q),$$

but also

$$\lim_{t \to -1^+} (1-t) f(t,q) = f_0(q),$$

where

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n},$$

one of Ramanujan's fifth order mock theta functions (cf. [30], [31]).

Elsewhere [2, p. 90–91], Andrews notes that if we take Σ to the be the Rogers-Ramanujan type series associated with Eq. (46) on Slater's list [27], and define its two variable generalization as

$$f(t,q) := \sum_{n=0}^{\infty} \frac{t^{3n} q^{n(3n-1)/2}}{(t;q)_{n+1}(t^2q;q^2)_n},$$

we find that $\lim_{t\to -1^+} f(t,q)$ is not a mock theta function, but rather a *false* theta function studied by Rogers [21, p. 333(2)].

Andrews later comments [2, p. 93]: "Now if we view this as a curve $y = f(t)$ the points of which are functions of q, we find that frequently if $f(1)$ is a modular form, $f(-1)$ is a mock or false theta function. Is there some general structure possible in which this seemingly amazing occurrence becomes more explicable?" While we do not have an answer to Andrews' question, we have observed that there is a third possibility. Namely, that $f(1)$ is a modular form and $f(-1)$ neither a mock nor false theta function, but rather a sum of modular forms.

Consider the following family of Rogers-Ramanujan type identities related to the modulus 27 which are due to Dyson [8, p. 433, Eqs. (B1)–(B4)] and reproved by Slater [27, p. 161–2, Eqs. (90)–(93)].

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}} = \frac{(q^{12},q^{15},q^{27};q^{27})_\infty}{(q;q)_\infty} \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} = \frac{(q^9;q^9)_\infty}{(q;q)_\infty} \tag{3.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}} = \frac{(q^6,q^{21},q^{27};q^{27})_\infty}{(q;q)_\infty} \tag{3.9}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}} = \frac{(q^3,q^{24},q^{27};q^{27})_\infty}{(q;q)_\infty} \tag{3.10}$$

The relevant two-variable generalizations (see [24, p. 15, Thm. 2.2]) of (3.7)–(3.10) are

$$f_{3.7}(t,q) := \frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{t^{2n}q^{n^2}(t^3q^3;q^3)_{n-1}}{(t;q)_{n+1}(t^2q;q)_{2n-1}} \tag{3.11}$$

$$f_{3.8}(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n}q^{n(n+1)}(t^3q^3;q^3)_n}{(t;q)_{n+1}(t^2q;q)_{2n+1}} \tag{3.12}$$

$$f_{3.9}(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n}q^{n(n+2)}(t^3q^3;q^3)_n}{(t;q)_{n+1}(t^2q;q)_{2n+2}} \tag{3.13}$$

$$f_{3.10}(t,q) := \sum_{n=0}^{\infty} \frac{t^{2n}q^{n(n+3)}(t^3q^3;q^3)_n}{(t;q)_{n+1}(t^2q;q)_{2n+2}}. \tag{3.14}$$

We believe that the four identities related to the modulus 108 recorded below, which arise from the $t = -1$ cases of (3.11)–(3.14), are new.

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q^3;q^3)_{n-1}}{(-q;q)_n(q;q)_{2n-1}} = \frac{(q^{12},q^{15},q^{27};q^{27})_\infty - 2q^2(-q^{33},-q^{75},q^{108};q^{108})_\infty + 2q^7(-q^{15},-q^{93},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.15}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^3;q^3)_{n}}{(-q;q)_n(q;q)_{2n+1}} = \frac{(q^{9},q^{18},q^{27};q^{27})_\infty - 2q^3(-q^{27},-q^{81},q^{108};q^{108})_\infty + 2q^9(-q^{9},-q^{99},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.16}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q^3;q^3)_{n}}{(-q;q)_n(q;q)_{2n+2}} = \frac{(q^{6},q^{21},q^{27};q^{27})_\infty - 2q^4(-q^{21},-q^{87},q^{108};q^{108})_\infty + 2q^{11}(-q^{3},-q^{105},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.17}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n+3)}(-q^3;q^3)_{n}}{(-q;q)_n(q;q)_{2n+2}} = \frac{(q^{3},q^{24},q^{27};q^{27})_\infty - 2q^5(-q^{15},-q^{93},q^{108};q^{108})_\infty + 2q^{13}(-q^{-3},-q^{111},q^{108};q^{108})_\infty}{(q;q)_\infty} \tag{3.18}$$

The identities (3.15)–(3.18) may be proved using Bailey pairs (see Sec. 2.3). Although less well known than (2.15), the following characterization of Bailey pairs [2, p. 29, Eq. (3.40) with $a = 1$] is equivalent to (2.15) with $a = 1$:

$$\alpha_n = (1-q^{2n})\sum_{k=0}^{n}\frac{(-1)^{n-k}q^{\binom{n-k}{2}}(q;q)_{n+k-1}}{(q;q)_{n-k}}\beta_k \tag{3.19}$$

Furthermore, it is straightforward to show that (3.19) may be rewritten as

$$\alpha_n = (-1)^n q^{\binom{n}{2}}(1+q^n)\sum_{k=0}^{n}(q^n;q)_k(q^{-n};q)_k q^k\beta_k, \tag{3.20}$$

which is the form we shall employ.

Lemma 3.1. *If, for n a nonnegative integer,*

$$\alpha_n = \begin{cases} 1 & \text{if } n=0 \\ (-1)^r q^{\frac{9}{2}r^2-\frac{3}{2}r}(1+q^{3r}) & \text{if } n=3r>0 \\ -2q^{18r^2+9r+1} & \text{if } n=6r+1 \\ 2q^{18r^2+15r+3} & \text{if } n=6r+2 \\ 2q^{18r^2+21r+6} & \text{if } n=6r+4 \\ -2q^{18r^2+27r+10} & \text{if } n=6r+5 \end{cases} \tag{3.21}$$

and

$$\beta_n = \begin{cases} \frac{(-q^3;q^3)_{n-1}}{(-q;q)_n(q;q)_{2n-1}} & \text{if } n>0 \\ 1 & \text{if } n=0, \end{cases} \tag{3.22}$$

then (α_n, β_n) *forms a Bailey pair.*

Proof. By inserting (3.21) and (3.22) into (3.20), it is clear that we will be done once we show

$$\begin{aligned}(-1)^r q^{\frac{9}{2}r^2-\frac{3}{2}r}(1+q^{3r}) \\ = (-1)^r q^{\binom{3r}{2}}(1+q^{3r})\left\{1+\sum_{k=1}^{3r}\frac{(q^{3r};q)_k(q^{-3r};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\},\end{aligned} \tag{3.23}$$

$$\begin{aligned}-2q^{18r^2+9r+1} \\ = -q^{\binom{6r+1}{2}}(1+q^{6r+1})\left\{1+\sum_{k=1}^{6r+1},\frac{(q^{6r+1};q)_k(q^{-6r-1};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\},\end{aligned} \tag{3.24}$$

$$\begin{aligned}2q^{18r^2+15r+3} \\ = q^{\binom{6r+2}{2}}(1+q^{6r+2})\left\{1+\sum_{k=1}^{6r+2}\frac{(q^{6r+2};q)_k(q^{-6r-2};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\},\end{aligned} \tag{3.25}$$

$$\begin{aligned}2q^{18r^2+21r+6} \\ = q^{\binom{6r+4}{2}}(1+q^{6r+4})\left\{1+\sum_{k=1}^{6r+4}\frac{(q^{6r+4};q)_k(q^{-6r-4};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\},\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}-2q^{18r^2+27r+10} \\ = -q^{\binom{6r+5}{2}}(1+q^{6r+5})\left\{1+\sum_{k=1}^{6r+5}\frac{(q^{6r+5};q)_k(q^{-6r-5};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}}\right\},\end{aligned} \tag{3.27}$$

for nonnegative integers r. Using elementary algebra, the equations (3.23)–(3.27) are easily shown to be equivalent to

$$\sum_{k=1}^{3r}\frac{(q^{3r};q)_k(q^{-3r};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 0, \tag{3.28}$$

$$\frac{q^{6r+1}+1}{q^{6r+1}-1}\sum_{k=1}^{6r+1}\frac{(q^{6r+1};q)_k(q^{-6r-1};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 1, \tag{3.29}$$

$$\frac{q^{6r+2}+1}{q^{6r+2}-1}\sum_{k=1}^{6r+2}\frac{(q^{6r+2};q)_k(q^{-6r-2};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 1, \tag{3.30}$$

$$\frac{1+q^{6r+4}}{1-q^{6r+4}}\sum_{k=1}^{6r+4}\frac{(q^{6r+4};q)_k(q^{-6r-4};q)_k(-q^3;q^3)_k q^k}{(-q;q)_k(q;q)_{2k-1}} = 1, \tag{3.31}$$

$$\frac{1+q^{6r+5}}{1-q^{6r+5}}\sum_{k=1}^{6r+5}\frac{(q^{6r+5};q)_k(q^{-6r-5};q)_k(-q^3;q^3)_kq^k}{(-q;q)_k(q;q)_{2k-1}}=1. \tag{3.32}$$

Each of equations (3.28)–(3.32) may be verified using the WZ method [18, Chapter 7] or induction on r. □

Theorem 3.2. *Identity* (3.15) *is valid.*

Proof. Recall that the weak form of Bailey's lemma [2, p. 27, Eq. (3.33) with $a=1$] states that

$$\sum_{n=0}^{\infty} q^{n^2}\beta_n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n \tag{3.33}$$

for any Bailey pair (α_n,β_n). Inserting the Bailey pair established in Lemma 3.1 into (3.33) yields

$$\begin{aligned}
&1+\sum_{n=1}^{\infty}\frac{q^{n^2}(-q^3;q^3)_{n-1}}{(-q;q)_n(q;q)_{2n-1}} \\
&\quad=\frac{1}{(q;q)_\infty}\left(1+\sum_{r=1}^{\infty}(-1)^r q^{\frac{27}{2}r^2-\frac{3}{2}r}(1+q^{3r})-2\sum_{r=0}^{\infty}q^{54r^2+21r+2}\right.\\
&\qquad\left.+2\sum_{r=0}^{\infty}q^{54r^2+39r+7}+2\sum_{r=0}^{\infty}q^{54r^2+69r+22}-2\sum_{r=0}^{\infty}q^{54r^2+87r+35}\right)\\
&\quad=\frac{1}{(q;q)_\infty}\left(\sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{27}{2}r^2-\frac{3}{2}r}-2\sum_{r=0}^{\infty}q^{54r^2+21r+2}+2\sum_{r=0}^{\infty}q^{54r^2+39r+7}\right.\\
&\qquad\left.+2\sum_{r=1}^{\infty}q^{18r^2-39r+7}-2\sum_{r=1}^{\infty}q^{54r^2-21r+2}\right)\\
&\quad=\frac{1}{(q;q)_\infty}\left(\sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{27}{2}r^2-\frac{3}{2}r}-2\sum_{r=-\infty}^{\infty}q^{54r^2-21r+2}+2\sum_{r=-\infty}^{\infty}q^{54r^2-39r+7}\right)\\
&\quad=\frac{(q^{12},q^{15},q^{27};q^{27})_\infty-2q^2(-q^{33},-q^{75},q^{108};q^{108})_\infty+2q^7(-q^{15},-q^{93};q^{108};q^{108})_\infty}{(q;q)_\infty}.
\end{aligned} \tag{3.34}$$

□

The other identities (3.16)–(3.18) may be proved similarly.

The identity (3.16) deserves special attention because its right hand side may be expressed as a single infinite product whereas it appears that of the other three can not be simplified beyond a sum of three infinite products.

Theorem 3.3.

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}}=\frac{(q^3;q^3)_\infty(q^3;q^{18})_\infty(q^{15};q^{18})_\infty}{(q;q)_\infty} \tag{3.35}$$

Proof. We shall require Fricke's quintuple product identity [12]

$$(z^3q,z^{-3}q^2,q^3;q^3)_\infty+z(z^{-3}q,z^3q^2,q^3;q^3)_\infty=(-z^{-1}q,-z,q;q)_\infty(z^{-2}q,z^2q;q^2)_\infty \tag{3.36}$$

and an identity due to Bailey [10, p. 220, Eq. (4.1)]

$$(3.37)\quad (-z^2q,-z^{-2}q^3,q^4;q^4)_\infty + z(-z^2q^3,-z^{-2}q,q^4;q^4)_\infty = (-z,-z^{-1}q,q;q)_\infty.$$

By (3.16), we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}}$$
$$= \frac{(q^9,q^{18},q^{27};q^{27})_\infty - 2q^3(-q^{27},-q^{81},q^{108};q^{108})_\infty + 2q^9(-q^9,-q^{99},q^{108};q^{108})_\infty}{(q;q)_\infty}.$$

Expanding the first triple product in the numerator by (3.37) with q replaced by q^{27} and $z=-q^9$ yields

$$\frac{(-q^{45},-q^{63},q^{108};q^{108})_\infty - 2q^3(-q^{27},-q^{81},q^{108};q^{108})_\infty + q^9(-q^9,-q^{99},q^{108};q^{108})_\infty}{(q;q)_\infty}.$$

Two further applications of (3.37) shows that the preceding expression is equal to

$$\frac{(-q^9,-q^{18},q^{27};q^{27})_\infty - q^3(-1,-q^{27},q^{27};q^{27})_\infty}{(q;q)_\infty}$$
$$= \frac{(q^3,q^6,q^9;q^9)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty}$$
$$= \frac{(q^3;q^3)_\infty(q^3,q^{15};q^{18})_\infty}{(q;q)_\infty},$$

where the penultimate equality follows from (3.36). □

We close this section by recalling that once a given Bailey pair is established, it may be utilized in connection with limiting cases of Bailey's lemma other than (3.33), thus yielding additional Rogers-Ramanujan type identities. For instance, if we insert the Bailey pair established in Lemma 3.1 into [2, p. 26, Eq. (3.28) with $n,\rho_1\to\infty$ and $\rho_2=-\sqrt{q}$], we obtain the identity related to the modulus 144:

$$(3.38)\quad 1+\sum_{n=1}^{\infty} \frac{q^{n^2}(-q;q^2)_n(-q^6;q^6)_{n-1}}{(-q^2;q^2)_n(q^2;q^2)_{2n-1}}$$
$$= \frac{(q^{15},q^{21},q^{36};q^{36})_\infty - 2q^3(-q^{42},-q^{102},q^{144};q^{144})_\infty + 2q^{10}(-q^{18},-q^{126},q^{144};q^{144})_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty}.$$

A partner of (3.38) is

$$(3.39)\quad \sum_{n=0}^{\infty} \frac{q^{n^2+4n}(-q;q^2)_{n+1}(-q^6;q^6)_n}{(-q^2;q^2)_n(q^2;q^2)_{2n+2}}$$
$$= \frac{(q^3,q^{33},q^{36};q^{36})_\infty - 2q^7(-q^{18},-q^{126},q^{144};q^{144})_\infty + 2q^{12}(-q^6,-q^{138},q^{144};q^{144})_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty}.$$

4. Concluding Remarks

For quite a long time we were convinced that there must exist a general transformation of the type found in Watson's theorem (see (2.11)), a transformation which would give the result in Theorem 2.5 as a special case for particular values of its parameters.

One reason we thought this transformation had to exist was the appearance of the $(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty$ term on the product side, which can be represented as an infinite series via the Jacobi triple product. This in turn brought to mind Watson's proof of the Rogers-Ramanujan identities, where he showed that these followed as special cases of (2.11).

However, we could not find such a transformation, but possibly our search was incomplete. Does the identity in Theorem 2.5 follow as a special case of some known transformation, perhaps some known transformation between basic hypergeometric series? Is this identity a special case of some as yet undiscovered general transformation?

As remarked at the end of the introduction, varying the form of the series S in (1.3) may lead to other new identities of the Rogers-Ramanujan-Slater type. In particular, one might hope for the discovery of new identities which are not readily proved within the framework of our present understanding of identities of the Rogers-Ramanujan-Slater type. We hope to continue these investigations in a subsequent paper.

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