Improved Augmented Matched Interface and Boundary (AMIB) Method for Solving Problems on Irregular 2D Domains

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Mathematical Models

- **Poisson Eqn. (time-indept.):**
  \[ \Delta u + ku = f(\bar{x}), \quad (1.1) \]

- **Boundary Condition:**
  \[ \alpha \Gamma u + \beta \Gamma \frac{\partial u}{\partial n} = \phi(\bar{x}), \quad (1.2) \]

- **Heat Eqn. (time-dept.):**
  \[ \frac{\partial u}{\partial t} = \beta \Delta u + g, \quad 0 \leq t \leq T, \quad (1.3) \]

- **Boundary Condition:**
  \[ \alpha \Gamma u + \beta \Gamma \frac{\partial u}{\partial n} = \psi(t, \bar{x}), \text{ on } \Gamma, \quad (1.4) \]

- **Initial Condition:**
  \[ u(0, \bar{x}) = u_0(\bar{x}), \quad (1.5) \]
Applications

Figure: Poisson–Boltzmann eqn. for electrostatic potential distribution over a protein.

Figure: Pennes Bioheat eqn. for heat dissipation in Magnetic Fluid Hyperthermia (MFH).
Interface Points, Fictitious Points, and Vertical Points
Fictitious Value Representations at Fictitious Points

\[
\tilde{u}_{FP} = \sum_{(x_I, y_J) \in S_{FP}} \tilde{w}_{I,J} u_{I,J} + \sum_{\bar{x}_{VP_i} \in V_{FP}} \tilde{w}_{VP_i} \phi(\bar{x}_{VP_i}), \quad \text{(2.1)}
\]

where \( S_{FP} \) is a set of chosen grid points and \( V_{FP} \) is a set of vertical points.
The Augmented System

\[
\begin{pmatrix}
A & B \\
C & I
\end{pmatrix}
\begin{pmatrix}
U \\
Q
\end{pmatrix} =
\begin{pmatrix}
F \\
\Phi
\end{pmatrix},
\]  

Let \( N_1 = \) number of interior grid points, \( N_2 = \) number of interface points, we have:

- \( A_{N_1 \times N_1} \)
- \( B_{5N_2 \times N_1} \)
- \( C_{N_1 \times 5N_2} \)
- \( I_{5N_2 \times 5N_2} \)
- \( U_{N_1 \times 1} \)
- \( Q_{5N_2 \times 1} \)

**Figure:** Nonzero entries of \( B \) and \( C \).
The ”starfish” Interface (Poisson Eqn.)

Figure: Numerical solution of the ”starfish” interface.

<table>
<thead>
<tr>
<th>$[N_x, N_y]$</th>
<th>$L^\infty$</th>
<th>$L^2$</th>
<th>BCG</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>error</td>
<td>order</td>
<td>iter no.</td>
</tr>
<tr>
<td>[65, 65]</td>
<td>1.91E-06</td>
<td></td>
<td>37</td>
</tr>
<tr>
<td>[129, 129]</td>
<td>1.19E-07</td>
<td>4.00</td>
<td>44</td>
</tr>
<tr>
<td>[257, 257]</td>
<td>5.01E-09</td>
<td>4.57</td>
<td>47</td>
</tr>
<tr>
<td>[513, 513]</td>
<td>2.86E-10</td>
<td>4.13</td>
<td>51</td>
</tr>
</tbody>
</table>
The ”butterfly” Interface (Heat Eqn.)

![Numerical solution of the ”butterfly” interface.](image)

**Figure:** Numerical solution of the ”butterfly” interface.

<table>
<thead>
<tr>
<th>$[N_x, N_y]$</th>
<th>$L^\infty$</th>
<th>$L^2$</th>
<th>BCG time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>error</td>
<td>error</td>
<td>order</td>
</tr>
<tr>
<td>[65, 65]</td>
<td>1.15E-04</td>
<td>1.09E-05</td>
<td>28</td>
</tr>
<tr>
<td>[129, 129]</td>
<td>5.39E-07</td>
<td>1.24E-07</td>
<td>69</td>
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<tr>
<td>[257, 257]</td>
<td>5.50E-09</td>
<td>1.38E-09</td>
<td>293</td>
</tr>
<tr>
<td>[513, 513]</td>
<td>3.09E-10</td>
<td>1.02E-10</td>
<td>1351</td>
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Table: Temporal convergence tests for solving the ImIBVP with the "butterfly"-shaped interface

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$L^\infty$ error</th>
<th>$L^2$ error</th>
<th>BCG time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>order</td>
<td>order</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.67E-03</td>
<td>9.24E-04</td>
<td>82</td>
</tr>
<tr>
<td>4</td>
<td>4.01E-04</td>
<td>2.23E-04</td>
<td>160</td>
</tr>
<tr>
<td>8</td>
<td>9.99E-05</td>
<td>5.54E-05</td>
<td>308</td>
</tr>
<tr>
<td>16</td>
<td>2.49E-05</td>
<td>1.38E-05</td>
<td>568</td>
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<tr>
<td>32</td>
<td>6.23E-06</td>
<td>3.46E-06</td>
<td>1104</td>
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<tr>
<td>64</td>
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<td>2038</td>
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<td>128</td>
<td>3.89E-07</td>
<td>2.16E-07</td>
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The "aircraft" Interface (Heat Eqn.)
Table: Convergence tests for solving the ImIBVP with the ”aircraft”-shaped interface of various scale factors

<table>
<thead>
<tr>
<th>scale factor $k$</th>
<th>no. of points</th>
<th>$L^\infty$</th>
<th>$L^2$</th>
<th>BCG time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IP</td>
<td>FP</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>662</td>
<td>909</td>
<td>5.24E-09</td>
<td>3.78E-10</td>
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<tr>
<td>1.3</td>
<td>856</td>
<td>1198</td>
<td>3.41E-09</td>
<td>2.48E-10</td>
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<tr>
<td>1.6</td>
<td>1060</td>
<td>1479</td>
<td>4.32E-09</td>
<td>3.16E-10</td>
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<td>1.9</td>
<td>1266</td>
<td>1765</td>
<td>3.43E-09</td>
<td>2.20E-10</td>
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</tbody>
</table>
Conclusion

Key characteristics of the developed AMIB method are:

- capable of solving problems over highly irregular domains
- capable of handling versatile boundary conditions
- unconditionally stable when solving time-dependent problems
- accelerated by the FFT for high efficiency
- fourth-order accuracy (in space)
References

Li, C., Zhao, S., Pentecost, B., Ren, Y., & Guan, Z. (2024). A fourth-order Cartesian grid method with FFT acceleration for elliptic and parabolic problems on irregular domains and arbitrarily curved boundaries. To be submitted.
