Solving Parabolic Interface Problems with a Finite Element Method

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Parabolic Interface PDEs

* Defined on a region which is split by an interface.
* Solutions may be discontinuous over the interface.

\[
\frac{\partial u}{\partial t} = \nabla \cdot (\alpha \nabla u) + f, \; x \in \Omega \\
u = g_D, \; x \in \partial \Omega_D \\
\alpha \frac{\partial u}{\partial n} = g_N, \; x \in \partial \Omega_N \\
[u] = u^+ - u^- = \phi, \; x \in \Gamma \\
[\alpha \frac{\partial u}{\partial n}] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi, \; x \in \Gamma
\]
Why Computational Methods?

- Difficultly/impossibility of finding closed form analytical solutions
- Handle a varying array problems without added complexity
- These methods have proven error bounds and convergence
Finite Element Methods (FEMs)

- Can use non-uniform meshes
- Piecewise polynomials over the elements using basis functions
- Projection theorems ensure that FEM finds the best approximation in the function space
Discontinuous Galerkin (DG) FEMs

- Allows for discontinuity over element boundaries
- Broken piecewise polynomials over the elements
- A natural means to apply the jump conditions in the parabolic interface problems
Creating a Conforming Triangulation

- Triangulation which conforms to the interface
- Triangulation can be refined, which gives room more accurate solution
First Test Problem (Starfish)

- Zero if not in the star
- Piecewise source term
- Initial Condition (Right)
Issues with the Test Problem

- Sinking behavior
- Trouble with the center
Changing the Test Problem

- Less complex geometry
- Initial Condition (Right)
Stability and Over-Penalizing the Interface

Penalty: 800
Penalty: 8000
Penalty: 80000
Penalty: 800000
Penalty: 8000000
Penalty: $8 \times 10^{13}$
Penalty Dependence on Mesh

- Penalty: 800000
- Optimally, no dependence on mesh
Elliptic Problem Over Mesh Refinement

* Keeps shape of exact solution
* Error around interface does not diminish
* Not optimal with Over-penalizing
Solving the Interfaced Heat Equation

- Sufficiently small time step
- Oscillation for larger time steps
- Stable in time without over-penalizing

Approximation and Error Plots
Discussion

- Method works for interface conditions equal to zero
- No success with non-zero interface conditions
- Over-penalizing required due to triangulation orientation
- Non-zero interface conditions (WG FEM, DDG FEM)
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Questions?