SOLVING THE HEAT EQUATION WITH INTERFACES

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Solving the Heat Equation with Interfaces

\[ u(x, y, t) = \begin{cases} 
  u^+(x, y, t) & \text{in } \Omega^+ \\
  u^-(x, y, t) & \text{in } \Omega^-
\end{cases} \]

\[ \frac{\partial u}{\partial t} = \nabla (\alpha \nabla u) + f \]

[\alpha u_n] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi

- \alpha - \text{diffusion coefficient}
- u - \text{function of interest}
- \Omega - \text{domain of interest}
- \Gamma - \text{interface}
- \Phi - \text{zeroth jump condition}
- \psi - \text{first jump condition}
Applications

• Metallurgy
  • Steel Continuous Casting

• Mathematical Biology
  • Cancer Treatment
  • Ecological Modeling
\[ u_{i,j}^k = u(x_i, y_j, t_k) \]
Temporal Discretization

- Euler Method
  - 1st Order Accuracy

\[
\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\alpha \Delta t} = \delta_{xx} u_{i,j}^{k+1} + \frac{f_{i,j}^{k+1}}{\alpha}
\]
Spatial Discretization

At regular nodes

\[ \delta_{xx} u_{i,j}^{k+1} = \frac{1}{\Delta x^2} (u_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1}) \]

At nodes adjacent to interface

\[ \delta_{xx} \tilde{u}_{i,j}^{k+1} = \frac{1}{\Delta x^2} (\tilde{u}_{i-1,j}^{k+1} - 2u_{i,j}^{k+1} + u_{i+1,j}^{k+1}) \]
Spatia\l Discretization

Transformations

\[ \frac{\partial}{\partial n} = \cos(\theta) \frac{\partial}{\partial x} + \sin(\theta) \frac{\partial}{\partial y} \]

\[ \frac{\partial}{\partial \tau} = -\sin(\theta) \frac{\partial}{\partial x} + \cos(\theta) \frac{\partial}{\partial y} \]

Jump Conditions

\([u] = u^+ - u^- = \phi\]

\([u_\tau] = \frac{\partial \phi}{\partial \tau} = \phi_\tau\]

\([\alpha u_x] = \psi \cos(\theta) - \sin(\theta)(\alpha^+ - \alpha^-)u^+_\tau - \sin(\theta)\alpha^- \phi_\tau = \bar{\psi}\]
Spatial Discretization

From \( \phi \) we have

\[
\begin{align*}
w^{+}_{0,1} \tilde{u}^{k+1}_{i,j} + w^{+}_{0,2} u^{k+1}_{i+1,j} + w^{+}_{0,2} u^{k+1}_{i+2,j} = \\
 w^{-}_{0,1} u^{k+1}_{i-1,j} + w^{-}_{0,2} u^{k+1}_{i,j} + w^{-}_{0,3} \tilde{u}^{k+1}_{i+1,j} + \phi
\end{align*}
\]

From \( \tilde{\psi} \) we have

\[
\begin{align*}
\alpha^{+} (w^{+}_{0,1} \tilde{u}^{k+1}_{i,j} + w^{+}_{0,2} u^{k+1}_{i+1,j} + w^{+}_{0,2} u^{k+1}_{i+2,j}) = \\
\alpha^{-} (w^{-}_{0,1} u^{k+1}_{i-1,j} + w^{-}_{0,2} u^{k+1}_{i,j} + w^{-}_{0,3} \tilde{u}^{k+1}_{i+1,j}) + \tilde{\psi}
\end{align*}
\]
Numerical Experiments

\[ \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial t^2} + f(x, t) \]

in \( \Omega = \Omega^+ \cup \Omega^- \)

Jump Conditions

\[ [u] = u^+ - u^- = \Phi \]

\[ [\alpha u_x] = \alpha^+ \frac{\partial u^+}{\partial x} - \alpha^- \frac{\partial u^-}{\partial x} = \psi \]

- \( \alpha \) – diffusion coefficient
- \( u \) – function of interest
- \( \Omega \) – domain of interest
- \( \Gamma \) – interface
- \( \Phi \) – function jump condition
- \( \psi \) – flux jump condition
Numerical Experiments

We will look at the example where the analytical solution is given as

\[ u^+(x, t) = \cos(x) \sin(t) \]
\[ u^-(x, t) = \sin(x) \cos(t) \]

With the following source terms

\[ f^+(x, t) = \alpha^+ \cos(x) \sin(t) + \cos(x) \cos(t) \]
\[ f^-(x, t) = \alpha^- \sin(x) \cos(t) - \sin(x) \sin(t) \]
Numerical Experiment

Result

Error
Numerical Experiments

We will look at the example where the analytical solution is given as

\[ u^+(x, t) = e^{-t} \sin(x) \]
\[ u^-(x, t) = e^{-t} \cos(x) \]

With the following jump conditions

\[ \phi = e^{-t} \sin(x) - e^{-t} \cos(x) \]
\[ \psi = 10e^{-t} \cos(x) + e^{-t} \sin(x) \]
Numerical Experiment

Result

Error
Future Improvements

• Increase to 2D
  • This will introduce additional complications at nodes near interfaces
• Increase complexity of Interfaces
• Utilize Peaceman-Rachford method
  • Higher accuracy in temporal discretization
• Address corner cases with irregular interface geometries
Conclusion

• This exercise demonstrates the effectiveness of MIB in solving the Heat Equation with Interfaces

• Further improvements are required for applications to real-world tasks