A Matched Alternative Direction Interface (ADI) Method For Solving Parabolic Interface Problems

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Abstract

Interface problems are a large class of problems that study the change of a physical quantity in Physics, Biology, Engineering or Materials, such as heat or electrostatic potential, as it propagates across a material interface. Due to the irregularly shaped interface, solutions to interface problems can only be found numerically. However, for the very same reason, classical numerical methods cannot deliver accurate estimations, or may fail entirely. A new numerical method is necessary for solving interface problems efficiently and accurately. In this project, we present our recent study of a well-tuned matched Alternative Direction Interface (ADI) method for solving two-dimensional interface problems with the most general of physical interface jump conditions. We also plan to present our recent improvements on the efficiency, accuracy, and stability of the proposed method.

Parabolic Interface Problems and Their Applications

The parabolic interface problems are mathematically described by:

\[
\frac{\partial u}{\partial t} = \nabla \cdot (\alpha \nabla u) + f \quad \text{in } \Omega = \Omega^+ \cup \Omega^-
\]

\[
[u] = u^+ - u^- = \phi(x, t), [\alpha u] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi(x, t).
\]

where \(u(x, t)\) is a function of interest, \(\alpha\) is the diffusion coefficient, and \(f\) is a source. Proper boundary conditions are prescribed on \(\partial \Omega\). The domain \(\Omega\) is split into two media \(\Omega^+\) and \(\Omega^-\) by a material interface \(\Gamma\). Across the interface \(\Gamma\), the diffusion coefficient \(\alpha\) is discontinuous, while the source term \(f\) may be even singular.

A Matched Alternative Direction Interface Method

\(\bullet\) **Temporal Discretization - Douglas ADI scheme**

\[
\frac{1}{\Delta t} (u_{i,j}^{k+1} - u_{i,j}^k) + \frac{\Delta t}{\alpha} \frac{\partial^2 u_{i,j}^k}{\partial x^2} = f_{i,j}^{k+1}
\]

\(\bullet\) **Spatial Discretization - Matched Interface and Boundary (MIB) method**

- Use the standard central difference formula on grids away from the interface

\[
\begin{align*}
\delta_{xx} u_{i,j}^k &= \frac{1}{\Delta x^2} (u_{i-1,j}^k - 2u_{i,j}^k + u_{i+1,j}^k) \\
\delta_{yy} u_{i,j}^k &= \frac{1}{\Delta y^2} (u_{i,j-1}^k - 2u_{i,j}^k + u_{i,j+1}^k)
\end{align*}
\]

- Incorporate the derived jump conditions

\[
\begin{align*}
[u] = \psi(x, t) - \sin \theta(u^+ - u^-), [\alpha u] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi(x, t)
\end{align*}
\]

- To correct the central difference formula on grids close to the interface

\[
\begin{align*}
\delta_{xx} u_{i,j}^k &= \frac{1}{\Delta x^2} (u_{i-1,j}^k - 2u_{i,j}^k + u_{i+1,j}^k) \\
\delta_{yy} u_{i,j}^k &= \frac{1}{\Delta y^2} (u_{i,j-1}^k - 2u_{i,j}^k + u_{i,j+1}^k)
\end{align*}
\]

Illustration of the MIB grid partitions. (a) For a regular interface. (b) For a corner point. In both figures, the jump conditions will be discretized by using fictitious (open circles) and function values (filled circles). In (a), the approximation of \(u_{i,j}^k\) is also shown, i.e., it is approximated by two auxiliary values (open squares), then interpolated by six function values (filled squares).

Recent Improvements

a) Change the temporal discretization formula from 1st order to 2nd order by replacing the Douglas ADI method with the Peaceman-Rachford ADI method while maintaining unconditional stability.

b) Improve the approximation of \(u_{i,j}^\tau\) by separately approximating tangent line \(\tau\) when it is exactly vertical or horizontal.

c) Make spatial approximations in both \(\Omega^+\) and \(\Omega^-\) by utilizing \(u_{i,j}^\tau\) to add in addition to \(u_{i,j}^\tau\).

References


[3] Blausen Medical Communications, Inc. (used under Creative Commons License)