JAMES MC LAUGHLIN'S RESEARCH STATEMENT

My research interests are mostly in the field of number theory. Currently, much of my work involves investigations in several areas which have connections to continued fractions.

1. The convergence of q-continued fractions on the boundary of regions of convergence

The most famous q-continued fraction is the Rogers-Ramanujan continued fraction,

$$K(q) = 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \ddots}}},$$

first studied by L.J. Rogers in 1894 and rediscovered by Ramanujan sometime before 1913. The convergence of $K(q)$ inside (via Worpitsky's Theorem) and outside the unit circle (a claim of Ramanujan proved in a paper of George Andrews, Bruce Berndt and others, [4]) is well understood. In 1917, I.Schur [57] completely settled the question of convergence at roots of unity. (Schur's result was also stated by Ramanujan, albeit with a small error) Schur's work shows that there is a dense set of points on the unit circle at which $K(q)$ converges and a dense set at which it diverges. Until the paper [15] Bowman and I wrote, it had been an open problem whether $K(q)$ converged or diverged at any points on the unit circle which are not roots of unity.

This paper [15] contained one of my thesis results, which was to show the existence of an uncountable set of points on the unit circle at which this continued fraction diverges and to give explicit examples of such points.

I have since proven similar results about convergence/divergence on the unit circle for a wider class of q-continued fractions, including several others also investigated by Ramanujan [16].

The question of convergence or divergence of the Rogers-Ramanujan continued fraction on a set of full measure on the unit circle is still open. It is conjectured that $K(q)$ diverges at every point on the unit circle that is not a root of unity. One current line of research is an attempt to prove this conjecture.

There are several variations/generalizations of $K(q)$ which have been investigated in the literature. In one generalization, which seems to have escaped attention, the sequence of exponents of q in the partial numerators is replaced by sequences other than $\{1, 2, 3, \dots\}$. A line of research I am pursuing is the investigation of the convergence behavior on the unit circle of such generalizations.

2. Other Convergence Questions for Continued Fractions

General convergence is a concept developed by Lisa Jacobsen ([28] and [29]) and is a generalization of ordinary convergence in that if a continued fraction converges, then it converges generally. In [15] I was able to prove the existence of an uncountable set of points on the unit circle where $K(q)$ does not converge in the general sense. In [17] I extended this result to a wider class of continued fractions, a class which includes various continued fractions investigated by Ramanujan and Selberg.

In [19] Douglas Bowman and I gave a new criterion for divergence in the general sense and then applied this criterion to two classes of q-continued fraction, to show that, for any particular q outside the unit circle, either the continued fraction converged or did not converge in the general sense.

In [44] Nancy Wyshinski and I gave a new convergence criterion for continued fractions of the form $K_{n=1}^{\infty} a_n/1$ and then applied this theorem to show the convergence of various classes of continued fraction, including some classes where $a_n \to \infty$.

In [46] Nancy Wyshinski and I used various techniques involving extensions and contractions of continued fractions to give new proofs of several results of Ramanujan.

One set of problems connected with convergence that I am currently investigating are questions relating to the separate convergence of the numerator and denominators of various q-continued fractions, the aim here being to give new proofs of some continued fraction identities of Ramanujan. Another line of investigation involves a recent discovery by Nancy Wyshinski and I that relates the Bauer-Muir transform of a continued fraction with a certain extension of the continued fraction. The aim is to select particular sequences for the Bauer-Muir transform that guarantees convergence of the extension and thus to derive some interesting continued fraction identities.

3. The regular continued expansion of various classes of real numbers

The problem of finding the regular continued fraction expansion of an irrational quantity expressed in some other form has a long history but until the 1970's not many examples of such continued fractions were known. The most familiar examples of such numbers are quadratic irrationals and various rational powers of e:

$$e^{2/(2n+1)} = [1; n, 12n+6, 5n+2,$$
$$\overline{1, 1, (6m+1)n+3m,\ (24m+12)n+12m+6,\ (6m+5)n+3m+2}\,]_{m=1}^{\infty}.$$

Lehmer [33] showed that certain quotients of modified Bessel functions, evaluated at various rationals, had a continued fraction expansion in which the partial quotients lay in an arithmetic progression.

A number of authors, including Böhmer [11], Mahler [37], Davison [23], Adams and Davison [1], Bowman [13], Borwein and Borwein [12], Shallit [60] and [61], Kmošek [30], Köhler [31] and Pethö [52] showed that the continued fraction expansion of various classes of numbers expressed as certain infinite series, had predictable partial quotients.

In [49], Mendès and van der Poorten considered a class of infinite products involving a variable X and showed that such products had a predictable continued fraction expansion in which all the partial quotients were polynomials in $\mathbb{Z}[x]$. Similar investigations, in which the continued fraction expansions of certain formal Laurent series is determined, can be found in [68], [67], [69] and [3].

In [63], Tamura investigated a particular infinite series involving the iterates of certain polynomials in $\mathbb{Z}[x]$ and showed, for all polynomials in a certain congruence class, that the continued fraction expansion of the series had all partial quotients in $\mathbb{Z}[x]$. In [21], Cohn gave a complete classification of all those polynomials $f(x) \in \mathbb{Z}[x]$ for which

$$\sum_{i=0}^{\infty} \frac{1}{f_i(x)}$$

had a regular continued fraction expansion in which all partial quotients were in $\mathbb{Z}[x]$. Here $f_i(x)$ is the i-th iterate of $f(x)$.

At the end of his paper Cohn listed a number of open questions and conjectures. One of the problems he mentioned was extending his results to the case of infinite products. In [41] I describe several infinite classes of polynomials $f(x) \in \mathbb{Z}[x]$ for which the infinite product

$$\prod_{i=0}^{\infty} \left(1 + \frac{1}{f_i(x)}\right) \tag{3.1}$$

has a regular continued fraction expansion with all partial quotients in $\mathbb{Z}[x]$. For all such polynomials, when the corresponding infinite product and its continued fractions are *specialized* by letting x be a positive integer, an irrational real number expressed as an infinite product is produced, a number whose regular continued fraction expansion has a predictable pattern. It is further shown that all such numbers are transcendental.

Another line of investigation into classes of real numbers with predictable continued fraction expansion was begun in [45], where Nancy Wyshinski and I showed how to manipulate various q-continued fraction identities of Ramanujan to produce infinite families of real numbers expressed as infinite products or quotients of various infinite series, where each such real number had a predictable continued fraction expansion. An example of one our results from this paper is the following.

For r, s and $q \in \mathbb{C}$ with $|q| < 1$, define

$$\phi(r,s,q) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2} r^n}{(q;q)_n(-sq;q)_n}.$$

Here $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$. Let $m > 1$ and $n > 2$ be positive integers and let d be rational such that $dn \in \mathbb{Z}^+$ and $dmn > 1$. Then

$$[1, \overline{d^{2k-2}n^{2k-1}, m^{2k-1}-1, 1, d^{2k-1}n^{2k}-2, 1, m^{2k}-1}]_{k=1}^{\infty} = \frac{\phi(-dm,-d,-1/(dmn))}{\phi(1/n,-d,-1/(dmn))}.$$

Future work will include completing the classification of all those polynomials $f(x)$ for which the infinite product at (3.1) has a specializable continued fraction expansion. Other work will include applying similar methods to other infinite series and infinite products, whose terms involve iterates of polynomials, to derive new infinite classes of irrational numbers whose regular continued fraction expansions have predictable partial quotients and to derive information on transcendence for these numbers.

On the q-continued fraction side, future work will include investigation of other q-continued fraction identities, including various transformations and manipulations of these, with the aim of producing further families of real numbers, expressed using infinite products or infinite series, whose regular continued fraction expansion has predictable patterns.

4. Polynomial Continued Fractions

Another area of my research is polynomial continued fractions. A polynomial continued fraction has the form $K_{n=1}^{\infty} f(n)/g(n)$ where $f(x)$, $g(x) \in \mathbb{Z}[x]$. Many well known constants like π and e have polynomial continued fraction expansions and the topic has a long history.

$$\frac{4}{\pi} = 1 + K_{n=1}^{\infty} \frac{(2n-1)^2}{2}.$$

However the general solution is known only in the cases where partial numerators have degree at most 2 and partial denominators have degree at most one (first examined by Oskar Perron [51]), where these continued fractions are evaluated in terms of hyper-geometric series. In [14], Douglas Bowman and I developed methods based on a theorem of Pincherle and a variant of the Euler transformation which allowed the limit of continued fractions in a large number of infinite families to be found. The degrees of the polynomials in these cases can be arbitrarily large. Using general theorems on continued fractions with irrational limits we showed that certain transcendental functions were irrational at certain rational arguments. In [47], Nancy Wyshinski and I extended some of this work (using extensions of known identities between continued fractions and infinite series and infinite products, extensions of continued fractions and the Bauer-Muir transform) to show how to produce infinite families of inequivalent polynomial continued fractions, each of which converged to the same limit.

The general problem of finding the limit of an arbitrary convergent polynomial continued fraction is still open. In [47] we raised several questions about the nature of the set of real numbers with polynomial continued fraction expansions. Future research in this area will include trying to answer some of these questions.

5. Polynomial Solutions to Pell's Equation

As several authors have remarked (for example, Alf van der Poorten in [65]), it remains an interesting problem to detect infinite families of positive integers D for which one can easily describe the fundamental unit in $\mathbb{Q}(\sqrt{D})$. In essence this involves finding polynomial solutions to Pell's equation, for which the square root of the polynomial giving an infinite family of values for D has an interesting and/or predictable continued fraction expansion. This also is an area which had seen little research since the investigations of Perron [51], until around 1976. Since then, there have been several interesting papers on the subject, including those by Bernstein [7], [8] and [9], Levesque and Rhin [34], van der Poorten [65], van der Poorten and Williams [66], Madden [36] and, most recently, Mollin and Goddard [50].

In [38], I showed how to construct several polynomial solutions to Pell's equation from each numerical solution and described the continued fraction expansion of the polynomial and the fundamental units in the corresponding quadratic fields. In [39], I showed how to construct, for each positive integer n, a finite collection of multi-variable polynomials such that each positive integer whose square root has a continued fraction expansion of period n is a value taken by one of these polynomials. Here also the continued fraction expansion of the square roots of the polynomials and the fundamental units in the corresponding infinite family of quadratic fields are given explicitly.

My future work in this area will include extending the work begun in [38] to polynomials of arbitrarily high degree and the investigation of the patterns that are possible in the continued fraction expansion of the square roots of these and other similar polynomials.

6. q-Continued fractions with multiple Limits

In a recent paper by Andrews et al ([2]), the authors exhibit a class of continued fraction in a variable z with three limit points. They also prove a statement of Ramanujan concerning a q-continued fraction with three limit points. Moreover, they remark that they know of no other similar q-continued fraction in the literature. In [20] Doug Bowman and I were able to generalize their class of continued fractions with three limit points and exhibit infinite classes of continued fractions in a variable z with arbitrarily many limit points. We were further able to give examples of infinite classes of q-continued fractions with arbitrarily many limit points, showing that

Ramanujan's q-continued fraction is not unique. We were also able to give a specific generalization of the Ramanujan q-continued fraction with three limit points to a q-continued fraction with $m \geq 3$ limit points and to evaluate these m limits as ratios of certain q-series.

The Ramanujan continued fraction with three limit points becomes a convergent continued fraction upon making the substitution $q \to 1/q$. This convergent continued fraction was also investigated by Ramanujan and has the nice property of converging to the infinite product $(q^2; q^3)_\infty/(q; q^3)_\infty$. As with the Ramanujan continued fraction with three limit points, our generalization has the property of becoming a convergent continued fraction upon making the substitution $q \to 1/q$. My future work in this area will include investigating the properties of this convergent continued fraction(derived from our generalization via the substitution $q \to 1/q$).

7. Other Work

I also have interests in many other projects, two of which are described below.

1. Combinatorial Identities: In [42] I used a new formula for the n-th power of a 2×2 matrix to give new proofs of some old combinatorial identities and to produce new identities. In [43] Nancy Wyshinski and I continued this work, finding more new combinatorial identities and new proofs for other known identities. Future work in this area will include extending the matrix methods developed in these papers further.

2. Rational Points on Elliptic Curves: The paper [27] was the result of summer research carried out with an undergraduate, student Saiying He. Let $p \geq 5$ be a prime and for $a, b \in \mathbb{F}_p$, let $E_{a,b}$ denote the elliptic curve over $\mathbb{F}_p$ with equation $y^2 = x^3 + a\,x + b$. As usual define the trace of Frobenius $a_{p,\,a,\,b}$ by

$$\#E_{a,b}(\mathbb{F}_p) = p + 1 - a_{p,\,a,\,b}.$$

We use elementary facts about exponential sums and known results about binary quadratic forms over finite fields to evaluate the sums $\sum_{t\in\mathbb{F}_p} a_{p,\,t,\,b}$, $\sum_{t\in\mathbb{F}_p} a_{p,\,a,\,t}$, $\sum_{t\in\mathbb{F}_p} a^2_{p,\,t,\,b}$, $\sum_{t\in\mathbb{F}_p} a^2_{p,\,a,\,t}$ and $\sum_{t\in\mathbb{F}_p} a^3_{p,\,t,\,b}$ for primes p in various congruence classes. Perhaps our most interesting result was the following: Let $p \equiv 5 \pmod 6$ be prime and let $b \in \mathbb{F}_p^*$. Then

$$\sum_{t\in\mathbb{F}_p} a^3_{p,\,t,\,b} = -p\left((p-2)\left(\frac{-2}{p}\right) + 2p\right)\left(\frac{b}{p}\right).$$

Future work in this area will include evaluating sums of the form $\sum_{t\in\mathbb{F}_p} a^k_{p,\,t,\,b}$ for odd $k \geq 5$.

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